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Interaction of electrons and an electromagnetic field treated by analytic continuation

By S. Bertil Nilsson

Summary. The analytic-continuation method ('α-method') for eliminating divergences has been found to be equivalent with the λ-limiting process in the classical theory of point charges, but to be more powerful in the quantum theory. The present paper contains an elaboration of the two forms in which the α-method has been introduced into quantum electrodynamics (by Gustafson and by Pauli) and further discusses their interrelation and the connection with the classical theory of Riesz and Fremberg and with the λ-process. The logarithmic divergences, which have so far been unaffected by the method, are now removed in the simplest cases (though not in the higher approximations); if the procedure is combined with Schwinger's method, the infinite mass-term that Schwinger isolates then becomes finite.

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Introduction

The main problem underlying the present thesis concerns the possibility of eliminating the well-known divergences in quantum electrodynamics by means of a special device of analytic continuation. This possibility was brought to the fore through the investigations of Riesz and Fremberg within the classical theory of point charges, where the infinities coming from the point singularity could be removed in a very natural manner with the help of the integration method for hyperbolic differential equations developed by Riesz in 1933—1936; the method involves an analytic continuation in a parameter α. The translation of the procedure into the quantum theory was undertaken, with good effect, by Gustafson in a series of papers during 1945 and 1946.

The present investigation forms a sequel to three earlier papers, which constitute the first parts of this thesis; they will here be referred to as [I], [II], and [III][1]. In [I] (1946) Gustafson's procedure is applied to a self-energy calculation in the hole theory; in [II] (1948) general consequences with respect to the elimination of divergences are established (in the second-order approximation) with account taken of the retardation, which had so far been neglected; and in [III] (1948) the approximation is pushed to the fourth order (still divergent), using a formulation due to Pauli. A short account of the results obtained will be given later, in sections 7 and 19.

Whereas [I—III] are mainly concerned with applications to various self-energy problems, the present paper aims at presenting a more general view of the situation, and in particular at elucidating the interrelation between the two forms in which the analytic-continuation method, or 'α-method', has been introduced into quantum electrodynamics (by Gustafson and by Pauli). To bring out also the connection with the classical theory of Riesz and Fremberg, a survey of their results is given in Chapter I, where the characterization of the procedure as an integration method is stressed. In Chapter II the application to quantum electrodynamics is made along similar lines in close connection with Gustafson's procedure; besides to self-energy problems attention is also paid to the question of the inner consistency of the definitions used. It turns out that certain complications, though probably not very serious ones, are caused by the fact of the formalism not being Hamiltonian for general values of the parameter α. A modification of the scheme intended to put it into a Hamiltonian form is expounded in Chapter III; the change required in the original Riesz kernel to achieve this objective deprives the theory of its relativistic aspect (except for the end value of the parameter.) The new formulation coincides in the main with that proposed by Pauli and is found to constitute a generalization — more powerful in eliminating divergences — of the well-known λ-limiting process. As a matter of fact the formulations of Gustafson and of Pauli had in [II] already been shown to remove all types of divergences except logarithmic ones. Chapter IV takes up the question of the remaining infinities again on the basis of a suggestion by Riesz, which is capable of eliminating the logarithmic divergences in e. g. the lowest approximation of the hole theory, but not those appearing in higher approximations. A concluding chapter deals briefly with a few miscellaneous subjects, such as relativistic invariance (combination with Schwinger's method), applications to meson theory, and other uses of analytic continuation.

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[1] See the alphabetical list of references at the end of the text.

The work on this paper was carried out at the Institute of Mechanics and Mathematical Physics in Lund. It is a pleasure to me to take this opportunity of expressing my gratitude to my teacher and the head of the Institute, Professor TORSTEN GUSTAFSON, who originally suggested this field of investigation to me. I am greatly indebted to him for his encouraging interest in my work and in my studies, which he has facilitated in various ways, for many valuable discussions and suggestions, and for much good advice.

I want to express my thanks to my teacher in mathematics, Professor MARCEL RIESZ, whose interest in fundamental physical questions of a mathematical nature has resulted in numerous stimulating discussions through many years, from which I have derived great advantage. In the present paper, Chapter IV was made possible through a suggestion of his, and I also want to thank him for criticizing part of the manuscript.

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Notations. It is convenient to collect here some notations of a general character that will be employed in the following. We work in the four-dimensional space-time of the restricted theory of relativity, in which a point $\boldsymbol{x}$ has four coordinates x_0, x_1, x_2, x_3 (or generally x_ν), x_0 standing for ct if t is the time and c the velocity of light. Alternatively, an imaginary time coordinate $x_4 = i\,c\,t$ may be used instead of x_0. In the same way a general four-vector is written $\boldsymbol{U} = U_\nu$ with $\nu = 0, 1, 2, 3$; or U_0 may be replaced by $U_4 = i\,U_0$. The space part of $\boldsymbol{U}$, in a special Lorentz frame, is $\vec{U} = U_k$ for $k = 1, 2, 3$.

To define the scalar product of two vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ we introduce the metrical tensor $g^{\mu\nu} = g_{\mu\nu}$ with the non-vanishing elements $g^{00} = -1$, $g^{11} = g^{22} = g^{33} = 1$. Then

$$(\boldsymbol{U}, \boldsymbol{V}) = g^{\mu\nu}\, U_\mu\, V_\nu = U^\nu\, V_\nu = U_\nu\, V^\nu, \tag{0.1}$$

where the covariant components $U^\nu = g^{\nu\mu} U_\mu$ are given by $U^0 = -U_0$, $U^k = U_k$ $(k = 1, 2, 3)$.[1] If nothing is said to the contrary, Einstein's summation convention is always understood to operate, Greek indices running from 0 to 3. A reader who should prefer imaginary time components may read all such summations as extending instead from 1 to 4, by putting $g^{44} = g_{44} = 1$, i. e. $U^4 = U_4$ $(= -i U^0)$; of course it is then unnecessary to distinguish between upper and lower indices. In general tensor expressions the passage from 0's to 4's, or conversely, is also immediate. If repeated Latin indices are summed over the values 1, 2, 3, the scalar product of two vectors in ordinary space is represented by $U_k V_k = \vec{U}\,\vec{V}$, so that $(\boldsymbol{U}, \boldsymbol{V}) = \vec{U}\,\vec{V} - U_0 V_0$. The vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ are said to be orthogonal if $(\boldsymbol{U}, \boldsymbol{V}) = 0$.

Following Riesz, we write $R(\boldsymbol{x}-\boldsymbol{y})$ for the square of the Lorentz distance between two points $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$R(\boldsymbol{x}-\boldsymbol{y}) = -(\boldsymbol{x}-\boldsymbol{y}, \boldsymbol{x}-\boldsymbol{y}). \tag{0.2}$$

According to this definition $R(\boldsymbol{x})$ is positive for a time-like vector $\boldsymbol{x}$ and negative for a space-like one. In particular, $R(\boldsymbol{x}) = 0$ is the equation of a cone in four dimensions, the light-cone, with its vertex at the origin.

The symbol $\Box$ signifies the d'Alembertian or wave operator

$$\Box = g_{\mu\nu} \frac{\partial^2}{\partial x_\mu \, \partial x_\nu} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_0^2}. \tag{0.3}$$

A four-dimensional volume element $d x_0 \, d x_1 \, d x_2 \, d x_3$ is written $(d x)^4$ for short, and similarly $d x_1 \, d x_2 \, d x_3 = (d x)^3$; the index, 3 or 4, may be left out if there is no risk of ambiguity. Corresponding notations will be used in other vector spaces.

A space-like three-dimensional hypersurface σ (or S) has a time-like normal direction at every point. If $\boldsymbol{n}$ stands for the unit normal $(n_\nu n^\nu = -1)$ for which $n_0 > 0$, we shall sometimes use the notation

$$d \sigma_\nu = -n_\nu \, d \sigma, \tag{0.4}$$

$d \sigma$ being the absolute value of the three-dimensional surface element. With σ a plane orthogonal to the time axis we have $d \sigma = d \sigma^0 = (d x)^3$; thus the 'surface integral' over σ is then an ordinary volume integral in three-dimensional space.

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[1] In view of the custom among many quantum physicists of using the notation $U_4 = i U_0$ (with lower indices), it was thought worth while to deviate from the usual convention in tensor calculus about the placing of the indices, and to denote the 'ordinary' (contravariant) vector components by subscripts, The only difference between the two kinds of components being here that $U^0 = -U_0$, this should not cause any confusion. (Cf. also the writings of Dirac.)

Chapter I

Classical theory

1. The point electron in classical electrodynamics. Even within the framework of classical electrodynamics, the motion of an electron in an external field presents a fundamental theoretical problem, which is linked up with the question of the inner structure of the electron. This comes from the fact that the force exerted by a charged particle on itself depends on the distribution of the charge. However, an approximate equation of motion could be derived by Lorentz[1] for a finite electron and, in the local rest system (i. e. when velocity $\vec{v}$ is just zero), takes the form

$$m \frac{d\vec{v}}{dt} = - e \vec{E} + \frac{2 e^2}{3 c^3} \frac{d^2 \vec{v}}{d t^2}. \tag{1.1}$$

The charge on the electron has here been set equal to $-e$, and structure-dependent terms proportional to the radius and higher powers of the radius have been neglected.

The mass m in Eq. (1.1) is made up of the purely mechanical mass m_0 and an electromagnetic mass-term, which is inversely proportional to the radius of the electron. An infinity will therefore appear if the radius goes to zero, as it would for a point charge; and because of certain difficulties connected with a finite extension of the elementary charge, it is in fact natural to picture the electron as being in principle a point.[2] In spite of the singularity Dirac could, however, make it plausible, by considerations involving the energy and momentum of the field, that the relativistic formulation of (1.1),

$$m c^2 \dot{u}_\mu = \frac{2 e^2}{3} (\ddot{u}_\mu - \dot{\boldsymbol{u}}^2 u_\mu) - e F^{\text{in}}_{\mu\nu} u^\nu, \tag{1.2}$$

should hold exactly for a point electron, within the limits of the classical theory.[3] Here $F^{\text{in}}_{\mu\nu}$ is the electromagnetic field tensor[4] of the external field, and $\boldsymbol{u} = \dot{\boldsymbol{z}}$ is the velocity four-vector ($u_\mu u^\mu = -1$), dots denoting differentiations with respect to s, the proper time multiplied by the velocity of light.

To obtain this equation of motion Dirac had again to make an arbitrary compensation of a singular term; but it is also possible to arrive at the equation without singularities, with the help of some suitable mathematical artifice. Thus Wentzel[5] had combined the Dirac—Fock—Podolsky many-time formalism with a certain averaging and limiting process, later simplified by Dirac[6]:

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[1] See Lorentz [1] §§ 27, 37, and Note 18; or cf. Heitler [1] § 4.

[2] Cf. Frenkel [1] pp. 526—528. A relativistic theory of an extended electron is, however, given in MacManus [1].

[3] Dirac [1], where some difficulties in the physical interpretation of the equation are, however, pointed out. For a discussion also of other possibilities, as well as for further references, se Eliezer [2] and Havas [1].

[4] If $\vec{E}$ and $\vec{H}$ are the electric and magnetic field strengths, then $F_{0k} = E_k$ and $F_{ik} = H_l$, i, k, l being an even permutation of 1, 2, 3; further $F_{\mu\nu} = - F_{\nu\mu}$.

[5] Wentzel [1].

[6] Dirac [2].

whereas the ordinary retarded field diverges on the world-line of the electron, the 'λ-limiting process' of these authors there leads to a finite expression, viz. half the difference of the retarded and advanced fields. That difference just gives rise to the first term on the right-hand side of (1.2). The same value on the world-line, and thus the same equation of motion, can be obtained very naturally by a process of analytic continuation founded on Riesz's method for solving hyperbolic differential equations[1], as applied in classical electrodynamics by Riesz and Fremberg.[2] It is this procedure that we are going to occupy ourselves with here.

Though the Riesz method uses quite a different line of approach from the λ-process, the methods have been shown by Ma[3] to be fully equivalent in the classical theory[4], in the sense that they yield identical expressions for all field quantities. According to both methods, incidentally, the mass m in Eq. (1. 2) is to be equated to the purely mechanical mass of the electron, a fact which corresponds to the electrostatic self-energy being zero, as is rather natural for an elementary particle.

2. The Riesz field. We start from Maxwell's equations written in terms of the electromagnetic potentials A_μ [5]:

$$\Box A_\mu(\boldsymbol{x}) = -\frac{1}{\varepsilon_0} s_\mu(\boldsymbol{x}), \tag{1.3}$$

$$\frac{\partial A_\mu(\boldsymbol{x})}{\partial x_\mu} = 0. \tag{1.4}$$

The numerical value of the constant ε_0 is $1/4\pi$ in ordinary Gaussian units — in Heaviside units it is 1 — if the current-density four-vector s_μ is normalized so as to make s_0 equal to the charge density. For a point electron $s_\mu(\boldsymbol{x})$ vanishes outside the world-line L of the particle:

$$s_\mu(\boldsymbol{x}) = -e\int_L \delta_4(\boldsymbol{x}-\boldsymbol{z})\, u_\mu\, ds = -e\, u_\mu\, \delta_3(\boldsymbol{x}-\boldsymbol{z}), \tag{1.5}$$

where, in the last expression, $\boldsymbol{x}-\boldsymbol{z}$ lies in a three-dimensional plane orthogonal to the world-line. As before, $\boldsymbol{z}$ is the position vector of the electron (charge $-e$) and $u_\mu = \dot{z}_\mu$; thus $u_\mu\, ds = dz_\mu$.

Assuming the current-distribution (i. e. the world-line) to be known, we are here interested in the special form under which $A_\mu(\boldsymbol{x})$ appears if Eq. (1. 3) is solved by means of Riesz's method. A few general remarks about the method may first be in place.

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[1] Riesz [1], [2]. See also on a particular question Fremberg [1] and for detailed accounts Fremberg [2] and Riesz [3].

[2] See Fremberg [2] Chap. 12, Fremberg [3], and Riesz [3] Chap. VI.

[3] Ma [1].

[4] Not in the quantum theory; see section 17.

[5] These are connected with the physically observable field strengths $F_{\mu\nu}$ by the well-known relation $F_{\mu\nu} = \partial A_\nu/\partial x^\mu - \partial A_\mu/\partial x^\nu$; in three-dimensional language A_0 is the scalar potential and $\vec{A}$ the vector potential of the field.

We define (with $R = -(\boldsymbol{x}, \boldsymbol{x})$)

$$V^{\alpha}(\boldsymbol{x}) = \begin{cases} \dfrac{1}{H(\alpha)} R(\boldsymbol{x})^{\frac{\alpha-4}{2}} & \text{for } R(\boldsymbol{x}) > 0,\ x_0 > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{1.6}$$

with

$$H(\alpha) = \pi\, 2^{\alpha-1}\, \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha-2}{2}\right). \tag{1.7}$$

With the help of V^{α} it is possible to form the solution $u(\boldsymbol{x})$ of

$$\Box\, u(\boldsymbol{x}) = -f(\boldsymbol{x}) \tag{1.8}$$

which, together with its normal derivative, takes prescribed initial values on a three-dimensional space-like surface S; according to Riesz it is obtained by analytic continuation in the parameter α to $\alpha = 0$ of the expression

$$u^{(\alpha)}(\boldsymbol{x}) = \int\limits_{\Omega_S} V^{\alpha+2}(\boldsymbol{x}-\boldsymbol{y})\, f(\boldsymbol{y})\, (dy)^4 + \\ + \int\limits_{S} \left[V^{\alpha+2}(\boldsymbol{x}-\boldsymbol{y}) \frac{d\,u(\boldsymbol{y})}{d\,n} - \frac{d\,V^{\alpha+2}(\boldsymbol{x}-\boldsymbol{y})}{d\,n} u(\boldsymbol{y}) \right] d\,S \tag{1.9}$$

($u^{(\alpha)}$ is first to be defined for such values of α as make the integrals converge; see the papers of Riesz and Fremberg cited in note 1, p. 374). Here $d/d\,n = n_{\nu}\,\partial/\partial y_{\nu}$ means the derivative along the inner normal $\boldsymbol{n}$ to the surface at the point $\boldsymbol{y}$, and Ω_s is the whole region of space-time on the future side of S. Since, by definition, the kernel $V^{\alpha}(\boldsymbol{x}-\boldsymbol{y})$ vanishes outside the retrograde light-cone from the point $\boldsymbol{x}$, the domain of integration is in reality only the interior[1] of that half-cone down to the surface S. In particular, under suitable conditions on f, S may recede into the infinite past.

As is easily verified, the kernel $V^{\alpha}(\boldsymbol{x})$ has the property that

$$\Box\, V^{\alpha+2}(\boldsymbol{x}) = -V^{\alpha}(\boldsymbol{x}). \tag{1.10}$$

The way (1.9) is built up ensures that the same relation then holds good also between $u^{(\alpha+2)}$ and $u^{(\alpha)}$.

If $A_{\mu}^{(\alpha)}$ is formed from Eq. (1.3) in accordance with this scheme, the volume integral reduces to a line integral owing to the singular character of s_{μ}. The formalism is developed in e. g. FREMBERG [2] Chap. 12; the rest of this section is only a summary of results that will be of importance in the following. With a slight change of the definition we write, in the absence of a boundary surface,

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[1] Moreover, the factor $1/H(2) = 0$ causes $V^2(\boldsymbol{x}-\boldsymbol{y})$ to vanish except for $R = 0$, and so the actual solution gets contributions only from the cone mantle (Huygens's principle).

$$A_\mu^{(\alpha)}(\boldsymbol{x}) = -\frac{e}{\varepsilon_0} l_0^{-\alpha} \int_L V^{\alpha+2}(\boldsymbol{x}-\boldsymbol{z})\, dz_\mu, \tag{1.11}$$

where l_0 is a constant with the dimension of a length, introduced to make $A_\mu^{(\alpha)}$ dimensionally independent of α.[1] The surface integrals corresponding to those in (1.9) have been assumed to vanish in the limit as S tends to infinity in the direction of negative x_0, which is true if the world-line approaches a time-like asymptote for $z_0 \to -\infty$. This being presumed, the integral over L converges for $0 < \alpha < 1$, and is defined for other values of α by analytic continuation. The simple form (1.11) also implies that there is no external field present; if there is an external field, or more than one electron, further terms have to be added on the right-hand side.

With Fremberg we use the notation 'Riesz potential' for the generalized potential vector $A_\mu^{(\alpha)}$. It is an analytic function of α, the quantity of physical importance being obtained for $\alpha = 0$.[2]

It is easy to establish that the general Riesz potential satisfies the divergence relation (1.4), and that

$$\Box A_\mu^{(\alpha)}(\boldsymbol{x}) = -\frac{1}{\varepsilon_0} s_\mu^{(\alpha)}(\boldsymbol{x}), \tag{1.12}$$

$s_\mu^{(\alpha)}$ being the Riesz current density

$$s_\mu^{(\alpha)}(\boldsymbol{x}) = -e l_0^{-\alpha} \int_L V^\alpha(\boldsymbol{x}-\boldsymbol{z})\, dz_\mu. \tag{1.13}$$

The Riesz field strengths $F_{\mu\nu}^{(\alpha)}$, obtained in the usual way by differentiation of $A_\mu^{(\alpha)}$, still obey Maxwell's equations with $s_\mu^{(\alpha)}$, as in (1.12), playing the role of the current density. Moreover, from the generalized energy-momentum tensor

$$T_{\mu\nu}^{(\alpha)} = \varepsilon_0 \left(F_{\mu\lambda}^{(\alpha)} F_\nu^{\ \lambda\,(\alpha)} - \tfrac{1}{4} g_{\mu\nu} F_{\lambda\sigma}^{(\alpha)} F^{\lambda\sigma\,(\alpha)} \right) \tag{1.14}$$

one arrives at a corresponding 'Lorentz force density' for general α's:

$$f_\mu^{(\alpha)} = -\frac{\partial T_{\mu\nu}^{(\alpha)}}{\partial x_\nu} = F_{\mu\nu}^{(\alpha)} s^{\nu\,(\alpha)}. \tag{1.15}$$

The singularity contained in the original formulation of the theory, Eqs. (1.3)—(1.5), has disappeared in the case of the general α-quantities $F_{\mu\nu}^{(\alpha)}$, $s_\mu^{(\alpha)}$, etc., for $\alpha > 0$; and it is interesting to see what happens when α tends to nought. First it may be remarked that, as they should, $A_\mu^{(\alpha)}$ and $F_{\mu\nu}^{(\alpha)}$ are found to coincide in the limit with the classical retarded fields of Liénard and Wiechert at points outside the world-line, and $s_\mu^{(\alpha)}$ to approach zero uniformly in every closed region that is free of electrons. But though the analytic continuation of $s_\mu^{(\alpha)}$ is in addition finite even on the world-line, the continuation of the integral of $s_\mu^{(\alpha)}$ over a three-dimensional domain that cuts the line orthogonally does not vanish; instead it equals $-e u_\mu$, the same value as is obtained on integrating the δ-function in (1.5). A further integral property, proved by Riesz[1], relates to the Lorentz force density $f_\mu^{(\alpha)}$: if Ω is a closed four-dimensional domain whose boundary surface intersects the world-line in $\boldsymbol{z}'$ and $\boldsymbol{z}''$ (where the surface is presupposed space-like, $s' < s''$), then

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[1] No such constant need appear explicitly in the formulas if these are regarded as relations between numbers (e.g., lengths to be measured in multiples of the unit l_0); so in [I] and [II] etc. Moreover, since α tends to zero in the end results, l_0 may safely be left out as unessential except in logarithmic expressions, cf. Chap. IV. (The logarithmic terms in [II] § IV conform to the choice $l_0 = 1/2\,m$.)

[2] The papers of Fremberg and Riesz, as well as [I] and [II], use a different normalization of the parameter in $A^{(\alpha)}$, the end value there being $\alpha = 2$. In comparing with e.g. FREMBERG [2] also note the difference in the metric and in the sign of the charge.

$$\operatorname*{cont}_{\alpha=0} \int_\Omega f_\mu^{(\alpha)}(\boldsymbol{x})\,(d\,x)^4 = -\,e \int_{s'}^{s''} F_{\mu\nu}(\boldsymbol{z})\,u^\nu\,d\,s\,, \tag{1.16}$$

where, on taking account also of a possible external field,

$$F_{\mu\nu}(\boldsymbol{z}) = \frac{e}{6\,\pi\,\varepsilon_0}(\ddot{u}_\mu\,u_\nu - \ddot{u}_\nu\,u_\mu) + F_{\mu\nu}^{\text{in}}(\boldsymbol{z})\,. \tag{1.17}$$

The continuation in (1.16) must be performed from values of $\alpha > 1$.

It is of some interest to note that the left-hand side in Eq. (1.16) comes out correct if $s_\mu^{(\alpha)}$ is again replaced in the integrand by the original expression s_μ comprising a δ-singularity; for as shown by Fremberg, the analytic continuation of $F_{\mu\nu}^{(\alpha)}$ calculated for a point $\boldsymbol{z}$ on the world-line yields precisely (1.17). Quite generally, all field quantities turn out finite on the world-line[2], even though they may take arbitrarily large values in the immediate neighbourhood (thus they are not continuous).

The Lorentz-Dirac equation of motion is now a simple consequence of the formalism. As a matter of fact one is led to it in two different ways. First, according to Fremberg's derivation of (1.17) as the field strength on the world-line, a point electron should experience the force $-\,e\,F_{\mu\nu}(\boldsymbol{z})\,u^\nu$, that is to say[3]

$$m\,c^2\,\dot{u}_\mu = -\,e\,F_{\mu\nu}(\boldsymbol{z})\,u^\nu\,, \tag{1.18}$$

which is only another mode of writing Eq. (1.2) (in the latter equation ε_0 has been put equal to $1/4\,\pi$). Secondly, since $f_\mu^{(\alpha)} = -\,\partial\,T_{\mu\nu}^{(\alpha)}/\partial\,x_\nu$, it is natural to interpret the negative of (1.16) as the flow of electromagnetic energy-momentum out through the whole three-dimensional surface of the region Ω, and then it follows from the conservation laws that

$$\Big[m\,c^2\,u_\mu\Big]_{s'}^{s''} = -\,e \int_{s'}^{s''} F_{\mu\nu}(\boldsymbol{z})\,u^\nu\,d\,s\,, \tag{1.19}$$

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[1] No proof is published. Cf. section 27.

[2] MA [1].

[3] Here and in the rest of the paper m stands for the mechanical mass.

where the expression on the left gives the change in mechanical energy and momentum. Obviously Eqs. (1.18) and (1.19) are identical.

3. **Interpretation of the formalism.** If we assume the formalism above to have a physical import, we may take the following view of the situation. Owing to the singular character of the point electron, the field $A_\mu(\boldsymbol{x})$ becomes singular too on the world-line, and so it is meaningless, for instance, to try to form the field derivatives necessary for determining the force on the electron. However, it is possible to give a unique definition of those derivatives, and generally of all field variables on the world-line as well as outside, by starting from the generalized field $A_\mu^{(\alpha)}$ that enters as an intermediate stage in Riesz's solution of the equations (the field on the world-line then appears as a sort of residuum). The values thus obtained being accepted, the problem is completely determined mathematically.

It is important that, as we have seen in (1.19), the results of this procedure are supported by considerations of energy and momentum. Ordinarily the usefulness of such considerations is impaired in the case of point charges by the infinities inherent in the quantities; but if the generalized stress tensor $T_{\mu\nu}^{(\alpha)}$ is used for the definitions, there will be no difficulties of that kind. The total energy (cP_0) and the total momentum (P_k) of the field are then to be defined as

$$P_\mu(\boldsymbol{z}) = \underset{\alpha=0}{\mathrm{cont}} \frac{1}{c} \int_\sigma T_{\mu\nu}^{(\alpha)}\, d\sigma^\nu \tag{1.20}$$

(convergence for $\alpha > 1/2$), σ being a space-like three-dimensional surface cutting the world-line in a point $\boldsymbol{z}$ (e. g., the plane $x_0 = z_0$). The notation $P_\mu(\boldsymbol{z})$ implies that the expression is a four-vector which depends on $\boldsymbol{z}$ but not on the particular shape of σ outside the world-line. That is true if there is no radiation field at infinitely distant points of space (the world-line must then approach a time-like asymptote for $z_0 \to -\infty$); otherwise we must write P_μ as a functional $P_\mu[\sigma]$ depending on the behaviour of σ at infinity, and then the energy and momentum of the field do not form a four-vector.

Anyway, in contrast with the ordinary expression not involving analytic continuation, (1.20) yields finite values, at least if some care is exercised in the evaluation; e. g. the integral should first be extended over a finite region whose boundary is then allowed to go to infinity after α is continued to nought (see also section 4). For an electron at rest in some Lorentz system one gets in this way

$$P_\mu = 0 \tag{1.21}$$

(cf. Eq. (1.23) below). More generally, it is seen from (1.16) and (1.19) that when we are allowed to write $P_\mu = P_\mu(\boldsymbol{z})$ the total momentum vector of field and particle is constant in time:

$$mc\dot{z}_\mu + P_\mu(\boldsymbol{z}) = \text{const.} \tag{1.22}$$

Of course there is much in the present formalism that is at variance with current conceptions. As an illustration a remark may here be inserted about a singularity of electromagnetic energy and momentum that may still be said to persist on the world-line in spite of our having apparently arrived at the results above without any singularities at all. Thus in the simple example of an electron at rest, the electromagnetic energy residing within a three-dimensional sphere of radius r with its centre on the electron is readily found to be[1]

$$W(r) = \operatorname*{cont}_{\alpha=0} \frac{\varepsilon_0}{2} \int\limits_{|\vec{x}| \le r} (\vec{E}^{(\alpha)})^2 (dx)^3 = -\frac{e^2}{8\pi\varepsilon_0 r} \tag{1.23}$$

in agreement with (1.21). For r tending to zero, $W(r)$ tends to minus infinity. This is the same term that Dirac[2] had to introduce by way of compensation in his discussion of energy and momentum balance, and was pictured by him as arising from an infinite negative mass at the centre of the electron; in the present treatment the singularity appears as a manifestation of the electromagnetic field.

Returning to the general Riesz field quantities we may note that, although Maxwell's equations are formally satisfied, we have no scheme involving equations of motion for general α's; and it is easy to see that it would not be very fruitful or natural to attempt such an extension.[3] Instead the Riesz field is to be regarded as a simple representation of the real field, with the property that as one passes back to the real field from it one gets at the same time mutually corroborating finite definitions of values on the world-line and of energy-momentum quantities. Those definitions seem to be the only reasonable ones *if* the electron is pictured as a point charge, and if one wishes to retain the ordinary field equations.

Laying the stress on the world-line values — in view of the subsequent developments —, we can say that the procedure amounts to treating the question of the electron's motion as a pure integration problem to be solved by the Riesz method: if we know the world-line of the electron up to a point $\boldsymbol{z}$, then the method, in conjunction with Eq. (1.18), enables us to continue the world-line beyond the point by supplying a definition of the field strength also at the point $\boldsymbol{z}$ itself.

4. **Modifications.** Pursuing the trend of the last paragraphs, we shall now make a few considerations which are not important at the present stage, but will be of some use later on (section 8, Chapter III). Of course it is physically immaterial how the field behaves for general α's, for example whether or no the Maxwell equations (1.12) are satisfied; what is important is the behaviour for α tending to nought. Provided the results are not altered, there

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[1] Use $E_k^{(\alpha)} = -\partial A_0^{(\alpha)}/\partial x_k$ with $A_0^{(\alpha)}(\boldsymbol{x}) = -\frac{e}{4\pi\varepsilon_0} r^{\alpha-1} f(\alpha)$, where $f(\alpha) = \frac{\alpha}{2} l_0^{-\alpha} \int\limits_0^\infty t^{\frac{\alpha}{2}-1} \cdot (t+1)^{-1/2} dt$ tends to unity as $\alpha \to 0$. This expression for $A_0^{(\alpha)}$ follows immediately from e. g. Eq. (1.31) (with $w_\alpha = l_0^{-\alpha}$) if R is set equal to $r^2 t$ (for an electron at rest we have $z_0 = x_0 - (R + r^2)^{1/2}$).

[2] Dirac [1] Eq. (21) and p. 155.

[3] In Chapter III we shall come back to a formulation for general α's, which will not be relativistic, however.

is then nothing to prevent us from modifying the expression (1.11) for $A_\mu^{(\alpha)}$ when $\alpha > 0$, should we find it convenient to do so. As a matter of fact, that expression is certainly the simplest and most natural one, but it may in some cases lead to purely formal difficulties for $\alpha > 0$ due to the contributions from the infinite parts of the world-line in (1.11).

With the procedure interpreted as an integration method for equation (1.3), two possible modifications are near at hand. In the first place $A_\mu^{(\alpha)}$ may be defined in accordance with (1.9), that is by not integrating over the whole world-line but only down to a three-dimensional surface S; to the line integral is then also added a surface integral over S, which, however, can be neglected in the absence of an external field if the end results are afterwards computed for S approaching $x_0 = -\infty$ (cf. the lines following Eq. (1.11)). With this definition the total electrostatic energy of an electron at rest, for example, comes out as zero automatically, in accordance with (1.21), no other precautions now having to be taken.

Secondly, we may modify the kernel V^α: since only the vicinity of the cone mantle plays a role for the solution, it is of no consequence for the integration problem if the form of the kernel is altered inside the light-cone for, say, $R > R_0$. More generally we may replace $V^{\alpha+2}$ in (1.11) by a new kernel defined with the help of some continuous weight function v_α (of dimension $l_0^{-\alpha}$):

$$\bar{V}^\alpha(\boldsymbol{x}) = V^{\alpha+2}(\boldsymbol{x})\, v_\alpha(\boldsymbol{x}), \tag{1.24}$$

i.e., writing $\dfrac{4\pi}{\alpha H(\alpha+2)}\, v_\alpha = w_\alpha$,

$$\bar{V}^\alpha(\boldsymbol{x}) = \frac{\alpha}{4\pi} R(\boldsymbol{x})^{\frac{\alpha-2}{2}}\, w_\alpha(\boldsymbol{x}) \quad \text{or} \quad 0, \tag{1.25}$$

according as $\boldsymbol{x}$ does or does not lie inside the direct light-cone from the origin, cf. (1.6). As before, $R = -(\boldsymbol{x}, \boldsymbol{x})$. It is apparent that the new kernel will lead to the same results as the old if we postulate the condition on v_α that it is to tend to unity, and its derivatives to zero, for $R \to 0$, $\alpha \to 0$. Since $1/H(\alpha+2) = \alpha/4\pi + O(\alpha^2)$[1], the same condition is then satisfied by w_α. The important point is that we have the possibility of letting v_α go down to zero for large values of R in such a way that no convergence problems arise at infinity. For example, we might choose $v_\alpha = l_0^{-\alpha}\, e^{-\alpha R/l_0^2}$; another simple choice is

$$\bar{V}_\alpha(\boldsymbol{x}) = l_0^{-\alpha}\, V_\varkappa^{\alpha+2}(\boldsymbol{x}) \quad \text{for} \quad \varkappa = \alpha/l_0, \tag{1.26}$$

$V_\varkappa^\alpha(\boldsymbol{x})$ being the Riesz kernel belonging to the equation $(-\Box + \varkappa^2)\, u = f$[2]:

$$V_\varkappa^\alpha = \sum_{k=0}^{\infty} \binom{-\alpha/2}{k} \varkappa^{2k}\, V^{\alpha+2k} = \frac{1}{\pi\, 2^{\frac{\alpha+2}{2}}\, \Gamma\left(\frac{\alpha}{2}\right)} \left(\frac{R^{\frac{1}{2}}}{\varkappa}\right)^{\frac{\alpha-4}{2}} J_{\frac{\alpha-4}{2}}(\varkappa R^{\frac{1}{2}}). \tag{1.27}$$

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[1] If $\beta = O(\alpha^n)$ as α goes to nought, then $|\beta| \leq C\alpha^n$.

[2] See RIESZ [2] p. 169 and [3] § 56*bis*, or FREMBERG [2] Chap. 11.

The relativistic character of the formalism for all α's is retained if v_α as a function of $\boldsymbol{x}$ depends only on R[1], as in the examples above. In Chapter III, however, we shall be obliged to renounce that requirement; thus in general

$$v_\alpha(\boldsymbol{x}) = v_\alpha(R, t_1, t_2, t_3), \tag{1.28}$$

the t_k's being three parameters characterizing the points on the hyperboloid surface $R = \text{constant}$. Nor will it be possible there to fulfil the above-mentioned condition on the derivatives of v_α, only that on v_α itself:

$$v_0(0, t_k) = 1. \tag{1.29}$$

Even then it is easy to see that the modified kernel is satisfactory for the integration problem at points outside the world-line. The generalized expression for the Riesz potential is

$$A_\mu^{(\alpha)}(\boldsymbol{x}) = -\frac{e}{\varepsilon_0}\int_L \bar{V}^\alpha(\boldsymbol{x}-\boldsymbol{z})\,dz_\mu, \tag{1.30}$$

or, with z_μ and t_k expressed as functions $z_\mu(\boldsymbol{x}, R), t_k(\boldsymbol{x}, R)$ of R and of the field point $\boldsymbol{x}$,

$$A_\mu^{(\alpha)}(\boldsymbol{x}) = \frac{e}{2\pi\varepsilon_0}\,\frac{\alpha}{2}\int_0^\infty R^{\frac{\alpha}{2}-1}\, w_\alpha(R, t_k)\,\frac{\partial z_\mu}{\partial R}\,dR. \tag{1.31}$$

As a consequence of our condition (1.29) on v_α or w_α, this expression gives[2], for α tending to nought,

$$A_\mu(\boldsymbol{x}) = \frac{e}{2\pi\varepsilon_0}\left\{\frac{\partial z_\mu}{\partial R}\right\}_{R=0}, \tag{1.34}$$

which is the ordinary retarded field[3]. As is immediately evident, corresponding results are obtained for all derivatives: they are not affected by the

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[1] A corresponding generalization is used in RIESZ [3] § 77.

[2] This follows from the lemma given in FREMBERG [2] § 13: $f(\alpha, t, y)$ being a continuous function of α, t, y for $0 < \alpha < A$, $0 < t < T$, and in the domain of variation of y, then

$$\lim_{\alpha\to 0} \alpha \int_0^T t^{\alpha-1} f(\alpha, t, y)\,dt = f(0, 0, y). \tag{1.32}$$

For other exponentials one has

$$\operatorname*{cont}_{\alpha=0} \alpha \int_0^T t^{\alpha-n}\,dt = 0 \qquad (n \neq 1). \tag{1.33}$$

In (1.32) the upper limit T may be infinite if the integral still converges for some value of α and continues to do so at the upper limit as $\alpha \to 0$.

[3] See FREMBERG [2] § 43, or RIESZ [3] § 76.

presence of w_α in (1.31), at points outside the world-line. We shall not now enter into the question of the values *on* the world-line.

It is interesting that the divergence of $A_\mu^{(\alpha)}$ still vanishes, for by direct differentiation of (1.30) under the sign of integration (which is now possible) we get

$$\frac{\partial A_\mu^{(\alpha)}(\boldsymbol{x})}{\partial x_\mu} = -\frac{e}{\varepsilon_0}\int\limits_L \frac{\partial \bar{V}^\alpha(\boldsymbol{x}-\boldsymbol{z})}{\partial x_\mu}\,d z_\mu = \frac{e}{\varepsilon_0}\int\limits_L \frac{\partial \bar{V}^\alpha(\boldsymbol{x}-\boldsymbol{z})}{\partial z_\mu}\,d z_\mu = 0, \qquad (1.35)$$

since $\bar{V}^\alpha$ is zero at both limits.

CHAPTER II

Quantum theory

5. Generalities. Field equations. As is well known, the divergence difficulties associated with the assumption of point charges remain unchanged in quantum electrodynamics. The specific quantum effects give rise to new divergences, which appear when one tries to solve the equations (at least if the ordinary perturbation method is used), typically manifesting themselves as an infinite contribution to the self-energy of the electron[1].

It is natural to inquire how the methods for avoiding divergences in the classical case work in the quantum-theoretical one. The λ-limiting process was developed parallelly in the two theories[2]; applied to quantum electrodynamics it results in elimination of the infinity that directly corresponds to the classical point singularity, the purely quantum-theoretical divergences still remaining. To remove them Dirac has proposed a theory in which there is also room for photons of negative energy besides the ordinary positive- energy ones[3]; the physical interpretation, however, is not very straightforward. As for analytic continuation, Riesz's integration method was introduced into quantum-electrodynamical calculations by Gustafson[4], who also found that it is then possible to do without the hypothesis of negative-energy photons. We intend here to look a little more closely into the possibility of extending the classical theory outlined in Chapter I to quantum electrodynamics.

In order to prepare for the application of Riesz's integration method, we shall first summarize a scheme of field equations in quantum electrodynamics.[5] The quantized electromagnetic field is described with the help of the vector potential $A_\mu(\boldsymbol{x})$, a quantized spinor field $\psi(\boldsymbol{x})$ being used to represent the electrons (as before the electronic charge is written $-e$). In the one-electron theory the equations can then be obtained from a Lagrangian density of the form

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[1] WALLER [1], OPPENHEIMER [1].

[2] WENTZEL [1], DIRAC [2], [4].

[3] DIRAC [3], [4], see also PAULI [3] and ELIEZER [2].

[4] GUSTAFSON [1] — [5]; see section 7 below.

[5] HEISENBERG—PAULI [1], FERMI [1], HEISENBERG [1]. See also e.g. PAULI [2] and the book WENTZEL [2].

$$\mathcal{L}(\boldsymbol{x}) = -\frac{\varepsilon_0}{2}\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A^\mu}{\partial x^\nu} - \hbar c\,\psi^\dagger\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+\frac{mc}{\hbar}\right)\psi - ie\,\psi^\dagger\gamma^\mu A_\mu\psi. \tag{2.1}$$

The γ_μ's are four-row matrices (components $\gamma_{\mu,\alpha\beta}$; $\alpha, \beta = 1, 2, 3, 4$) with the property

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\,g_{\mu\nu} \tag{2.2}$$

(thus γ_1, γ_2, γ_3, and $\gamma_4 = i\gamma_0$, are Hermitian); they are connected with the ordinary Dirac matrices α_μ, β (with $\alpha_0 = 1$) according to[1]

$$\gamma_\mu = -i\beta\alpha_\mu, \quad \text{thus} \quad \gamma_4 = \beta. \tag{2.3}$$

The complex conjugate spinor to $\psi = \psi_\alpha$ is written ψ^*, from which $\psi^\dagger$ is obtained as $\psi^\dagger = \psi^*\gamma_4 = \psi^*\beta$ (i.e. $\psi^\dagger_\alpha = \psi^*_\beta\beta_{\beta\alpha}$ with summation convention); then $\psi^\dagger\psi$ is relativistically invariant and $\psi^\dagger\gamma_\mu\psi = -i\psi^*\alpha_\mu\psi$ is a four-vector. Hence $\mathcal{L}$ is invariant.

The variational principle $\delta\int\mathcal{L}\,(dx)^4 = 0$ yields the field equations

$$\Box A_\mu(\boldsymbol{x}) = \frac{ie}{\varepsilon_0}\psi^\dagger(\boldsymbol{x})\,\gamma_\mu\psi(\boldsymbol{x}), \tag{2.4}$$

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+\frac{mc}{\hbar}\right)\psi(\boldsymbol{x}) = -\frac{ie}{\hbar c}A_\mu(\boldsymbol{x})\,\gamma^\mu\psi(\boldsymbol{x}). \tag{2.5}$$

Eq. (2.5) is equivalent to

$$\frac{\partial\psi^\dagger}{\partial x_\mu}\gamma_\mu - \frac{mc}{\hbar}\psi^\dagger = \frac{ie}{\hbar c}\psi^\dagger\gamma^\mu A_\mu. \tag{2.5'}$$

The quantum conditions for the fields are expressed by means of the commutation relations (with $\boldsymbol{x}$ written as $\vec{x}$, x_0)

$$\left[A_\mu(\vec{x}, x_0), \frac{\partial A_\nu(\vec{x}\,', x_0)}{\partial x_0}\right] = \frac{i\hbar c}{\varepsilon_0}g_{\mu\nu}\,\delta(\vec{x}-\vec{x}\,'), \tag{2.6}$$

$$\{\psi_\alpha(\vec{x}, x_0),\ \psi^*_\beta(\vec{x}\,', x_0)\} = \delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}\,'), \tag{2.7}$$

which are compatible with the field equations. Other combinations of the quantities entering in (2.6) and (2.7) commute or anticommute in a well-known manner (for the same time coordinate). The bracket symbols have the following meaning:

$$[A, B] = AB - BA, \quad \{A, B\} = AB + BA. \tag{2.8}$$

The state of the system is characterized by a normalized state vector $C = |\,\rangle$, which in the representation used here (the Heisenberg picture) is constant in time; expectation values are then formed according to the pattern $\langle F \rangle = \langle | F | \rangle = (C, FC)$. The formalism is completed by the requirement that the state vector should satisfy the auxiliary condition

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[1] For brevity we use the common notation α_μ with $\mu = 0, 1, 2, 3$ also for the α-matrices, though they do not form a four-vector. — The different uses of α etc. should not be too confusing.

$$\frac{\partial A_\mu(\boldsymbol{x})}{\partial x_\mu} | \rangle = 0. \tag{2.9}$$

Finally we write down the definition of the canonical energy-momentum tensor $T_{\mu\nu}$:

$$T_{\mu\nu} = -\sum_q \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial q}{\partial x_\nu} \right)} \frac{\partial q}{\partial x^\mu} + g_{\mu\nu} \mathcal{L}, \tag{2.10}$$

the summation being extended over the field variables q. In the present case, after symmetrizing the first term with respect to the order of the factors,

$$T_{\mu\nu} = \frac{\varepsilon_0}{2} \left[\left\{ \frac{\partial A_\lambda}{\partial x^\mu}, \frac{\partial A^\lambda}{\partial x^\nu} \right\} - g_{\mu\nu} \frac{\partial A_\lambda}{\partial x_\sigma} \frac{\partial A^\lambda}{\partial x^\sigma} \right] + \hbar c \, \psi^\dagger \gamma_\nu \frac{\partial \psi}{\partial x^\mu}. \tag{2.11}$$

The conservation laws find their expression in the fact that

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = 0. \tag{2.12}$$

6. **Introduction of the α-method.** In Gustafson's treatment of the system (2.4), (2.5) he had recourse to perturbation theory, i.e. expansion in powers of the dimensionless constant $(4\pi\varepsilon_0)^{-\frac{1}{2}} (e^2/\hbar c)^{\frac{1}{2}} \approx 1/\sqrt{137}$, analytic continuation being used to solve the first-order equations (with the right-hand side proportional to the perturbation parameter). In practice a perturbation method seems to be the only feasible way at present. However, to see the connection with the classical case let us first consider the exact form of the equations.

The right-hand side of Eq. (2.4) may be written $\frac{e}{\varepsilon_0} \psi^* (\boldsymbol{x}) \, \alpha_\mu \, \psi (\boldsymbol{x})$. Now $\psi^* (\boldsymbol{x}) \psi (\boldsymbol{x})$ is an operator with the characteristic values $\sum_n \delta(\vec{x} - \vec{z}_n)$[1], and so the connection with the classical point singularity is evident; by analogy we can hope that Riesz's method will be able to remove the divergences that are a consequence of that singularity. As before we regard the application of the method as a pure integration problem analogous to that formulated at the end of section 3: suppose we know the electron field ψ up to some space-like surface σ (e.g. up to the time x_0), then the problem is to indicate how the field is to be continued beyond the surface.

To imitate the classical procedure we have first to form the Riesz potential $A_\mu^{(\alpha)}$, which is defined up to σ:

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[1] See HEISENBERG-PAULI [1] II, p. 188.

$$A^{(\alpha)}_{\mu}(\boldsymbol{x}) = -\frac{i\,e}{\varepsilon_0}\, l_0^{-\alpha} \int V^{\alpha+2}(\boldsymbol{x}-\boldsymbol{y})\,\psi^{\dagger}(\boldsymbol{y})\,\gamma_{\mu}\,\psi(\boldsymbol{y})\,(dy)^4 + \mathcal{A}^{(\alpha)}_{\mu}(\boldsymbol{x}), \qquad (2.13)$$

the second term corresponding to the surface integral in (1.9) (cf. below). We have then to perform the analytical continuation to $\alpha = 0$ for $\boldsymbol{x}$ a point on the surface σ, the result to be introduced into the equation of motion (2.5). This would be in complete analogy with the way in which the classical equation of motion (1.18) was obtained; the question is, however, will this procedure in the quantum-theoretical case have any different effects from the ordinary procedure?

At first sight it would seem to give the same results, because the singularity in (2.13) is not apparent; the analytical continuation of $A^{(\alpha)}_{\mu}$ can be carried out formally as usual, and then yields an expression in the form of a retarded potential[1] just as the ordinary treatment (cf. also Eq. (2.22) for $\alpha = 0$). However, if $A^{(\alpha)}_{\mu}$ is characterized by its effect on the things it operates on (state vectors, other operators), the analytic continuation will in fact have other properties in some respects than the formal expression for $\alpha = 0$, as we shall see in the next few sections. To bring out the importance of the procedure we write, instead of (2.5),

$$\left(\gamma_{\mu}\,\frac{\partial}{\partial\, x_{\mu}} + \frac{m\,c}{\hbar}\right)\psi(\boldsymbol{x}) = -\frac{i\,e}{\hbar\,c}\, A^{(\alpha)}_{\mu}(\boldsymbol{x})\,\gamma^{\mu}\,\psi(\boldsymbol{x}) \qquad (2.14)$$

(to be continued analytically to $\alpha = 0$), as a natural counterpart[2] to the classical equation of motion (1.18).

Classically the method led also to finite definitions of energy and momentum quantities. The quantum analogue would be to define such quantities from a tensor $T^{(\alpha)}_{\mu\nu}$ formed as in (2.11) but with $A^{(\alpha)}_{\mu}$ taking the place of A_{μ} and with ψ obeying Eq. (2.14). Thus we get, for example, the following expression for the 'Hamiltonian' $H = c\,P_0$ (σ being taken as a plane $x_0 = \text{const.}$):

$$\begin{aligned} H = \operatorname*{cont}_{\alpha=0} \int \Bigg[& \frac{\varepsilon_0}{2}\left(\frac{\partial A^{(\alpha)}_{\lambda}}{\partial\, x_0}\,\frac{\partial A^{\lambda(\alpha)}}{\partial\, x_0} + \frac{\partial A^{(\alpha)}_{\lambda}}{\partial\, x_k}\,\frac{\partial A^{\lambda(\alpha)}}{\partial\, x_k}\right) + \\ & + \psi^{*}\left(\frac{\hbar\,c}{i}\,\alpha_k\,\frac{\partial}{\partial\, x_k} + m c^2\,\beta\right)\psi + e\,\psi^{*}\,\alpha^{\mu}\,A^{(\alpha)}_{\mu}\,\psi\Bigg]\,(dx)^3. \end{aligned} \qquad (2.15)$$

It is not sufficient, however, to lay down new definitions; in order to be satisfactory they must also be compatible with the equations of motion. That means here that the analytic continuation of the volume integral of $\partial\, T^{(\alpha)}_{\mu\nu}/\partial x_{\nu}$ over any four-dimensional region must be required to vanish as a consequence of (2.13) and (2.14):

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[1] Cf. e.g. FREMBERG [2] Eq. (12.9).

[2] On the connection between the Dirac equation and the equations of motion in classical relativistic point-mechanics see PAULI [1].

$$\operatorname*{cont}_{\alpha=0}\int\limits_{\Omega}\frac{\partial T^{(\alpha)}_{\mu\nu}}{\partial x_\nu}(d\,x)^4=0. \tag{2.16}$$

It is easy to establish that

$$\frac{\partial T^{(\alpha)}_{\mu\nu}}{\partial x_\nu}=\frac{\varepsilon_0}{2}\left\{\Box\,A^{\nu(\alpha)},\frac{\partial A^{(\alpha)}_\nu}{\partial x^\mu}\right\}-i\,e\,\psi^\dagger\gamma^\nu\frac{\partial A^{(\alpha)}_\nu}{\partial x^\mu}\psi. \tag{2.17}$$

If the analogy with the classical case could be pushed further, we should expect the continuation of the integral of (2.17) to be unaffected by the substitution of $i\,e\,\psi^\dagger\gamma^\nu\psi$ for $\varepsilon_0\,\Box\,A^{\nu(\alpha)}$ in the first term on the right-hand side of (2.17). This may seem promising, since the second term, apart from the order of the factors, is the negative of $\frac{i\,e}{2}\left\{\psi^\dagger\gamma^\nu\psi,\frac{\partial A^{(\alpha)}_\nu}{\partial x^\mu}\right\}$; but it is evident that an explicit calculation is needed to verify whether (2.16) is really fulfilled. We shall come back to such a calculation in section 9, in the lowest significant order of approximation, after first occupying ourselves with the important question whether the new definition of energy and momentum gives finite results.

Before going over to perturbation methods we shall however say a few words about the second term $\mathcal{A}^{(\alpha)}_\mu$ in (2.13). If a boundary surface S is used, as in the second paragraph of section 4, $\mathcal{A}^{(\alpha)}_\mu$ is a surface integral of the form given in (1.9). It is not true, however, that this integral tends to zero when S recedes into the infinite past, at it did in the classical case; the reason is that in quantum electrodynamics we have to do with a vacuum field $\mathcal{A}_\mu$, with $\Box\,\mathcal{A}_\mu=0$, satisfying the commutation relations (2.6) even in the absence of charges. The contribution of the vacuum field to the surface integral is

$$\mathcal{A}^{(\alpha)}_\mu(\boldsymbol{x})=l_0^{-\alpha}\int\limits_{S}\left[V^{\alpha+2}(\boldsymbol{x}-\boldsymbol{y})\frac{d\,\mathcal{A}_\mu(\boldsymbol{y})}{d\,n}-\frac{d\,V^{\alpha+2}(\boldsymbol{x}-\boldsymbol{y})}{d\,n}\mathcal{A}_\mu(\boldsymbol{y})\right]d\,S, \tag{2.18}$$

which we still write as $\mathcal{A}^{(\alpha)}_\mu$, supposing the rest of the surface integral to be treated together with the volume integral in (2.13).

7. **Gustafson's method. Applications to self-energy problems.** In quantum-electrodynamical calculations the equations have to be treated by some approximation method. As a rule the possibility of an expansion

$$\psi=\psi^{(0)}+\psi^{(1)}+\psi^{(2)}+\cdots \tag{2.19}$$

is assumed, $\psi^{(n)}$ being proportional to the nth power of the perturbation parameter ($\sim e$). According to (2.13) $A^{(\alpha)}_\mu$ will then be represented by a series of the same form in which $A^{(\alpha)(0)}_\mu=\mathcal{A}^{(\alpha)}_\mu$; $A^{(\alpha)(1)}_\mu$ is obtained as the volume integral in (2.13) with $\psi^{\dagger(0)}\gamma_\mu\psi^{(0)}$ in the integrand, etc. In this way we arrive at Gustafson's formulation.

Using $\hbar=c=1$ we can write $\psi^{(0)}$ as

$$\psi^{(0)}(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \sum_n \int a_n(\vec{p})\, u_n(\vec{p})\, e^{i(\boldsymbol{p}_n, \boldsymbol{x})} (d\,p)^3 \tag{2.20}$$

with $(\boldsymbol{p}, \boldsymbol{p}) = -m^2$ and

$$\{a_n(\vec{p}),\, a_m^*(\vec{p}\,')\} = \delta_{nm}\,\delta(\vec{p} - \vec{p}\,'). \tag{2.21}$$

The index n labels the four states belonging to a fixed momentum $\vec{p}$ ($n = 1, 2$ corresponding to positive energy and $n = 3, 4$ to negative); $u_n(\vec{p})$ is the corresponding Dirac spinor. Then $A_\mu^{(\alpha)(1)}$ is obtained in the form (cf. Eq. (2.28))

$$A_\mu^{(\alpha)(1)}(\boldsymbol{x}) =$$

$$= -\frac{e}{\varepsilon_0} \frac{l_0^{-\alpha}}{(2\pi)^3} \sum_n \sum_m \iint a_m'^*\, a_n\, (u_m'^*\, \alpha_\mu\, u_n) \frac{e^{i(\boldsymbol{p}_n - \boldsymbol{p}_m', \boldsymbol{x})}}{(\boldsymbol{p}_n - \boldsymbol{p}_m',\, \boldsymbol{p}_n - \boldsymbol{p}_m')^{1+\alpha/2}} (d\,p)\, d\,p'), \tag{2.22}$$

where we have used the abbreviated notation $u_n = u_n(\vec{p})$, $u_n' = u_n(\vec{p}\,')$, etc. The Fourier resolution of the vacuum field $A_\mu^{(0)} = \mathcal{A}_\mu$ is

$$\mathcal{A}_\mu(\boldsymbol{x}) = \frac{(2\pi)^{-3/2}}{\sqrt{2}} \int [A_\mu(\boldsymbol{k})\, e^{i(\boldsymbol{k}, \boldsymbol{x})} + A_\mu^*(\boldsymbol{k})\, e^{-i(\boldsymbol{k}, \boldsymbol{x})}] \frac{(d\,k)^3}{k}, \tag{2.23}$$

where $k_0 = |\vec{k}| = k$ and

$$[A_\mu(\boldsymbol{k}),\, A_\nu^*(\boldsymbol{k}')] = g_{\mu\nu} \frac{k}{\varepsilon_0}\, \delta(\vec{k} - \vec{k}\,'). \tag{2.24}$$

The commutation relations (2.24) and (2.21) are equivalent with (2.6) and (2.7).

The solution $\psi^{(1)}$ of the Dirac equation

$$\left(-i\,\alpha_\nu \frac{\partial}{\partial x_\nu} + \beta\, m\right) \psi^{(1)} = -e\, \mathcal{A}_\nu\, \alpha^\nu\, \psi^{(0)} \tag{2.25}$$

is obtained from (2.20) and (2.23) as

$$\psi^{(1)}(\boldsymbol{x}) =$$

$$= -e \frac{(2\pi)^{-3}}{\sqrt{2}} \sum_n \int (d\,p)^3 \int \frac{(d\,k)^3}{k} a_n(\vec{p}) \Bigg[A_\nu(\boldsymbol{k}) \frac{p_{n0} + k + \vec{\alpha}(\vec{p} + \vec{k}) + \beta\, m}{2(\boldsymbol{p}_n, \boldsymbol{k})} e^{i(\boldsymbol{p}_n + \boldsymbol{k}, \boldsymbol{x})} +$$

$$+ A_\nu^*(\boldsymbol{k}) \frac{p_{n0} - k + \vec{\alpha}(\vec{p} - \vec{k}) + \beta\, m}{-2(\boldsymbol{p}_n, \boldsymbol{k})} e^{i(\boldsymbol{p}_n - \boldsymbol{k}, \boldsymbol{x})} \Bigg] \alpha^\nu\, u_n(\vec{p}) \tag{2.26}$$

(for $\alpha = 0$).

The effect of using analytic continuation can best be illustrated in the example of the *self-energy of the electron*. We shall here give a summary of results obtained in the lowest order of approximation that does not vanish identically, i. e. terms proportional to e^2. By analogy with the ordinary expression[1] Gustafson defines this approximation of the self-energy as the expectation value of half the interaction term in (2.15), or, if for the moment we leave the order of the factors out of account,

$$W = \operatorname*{cont}_{\alpha=0} \frac{e}{2} \int \psi^* \alpha^\mu \psi A_\mu^{(\alpha)} (d x)^3 \tag{2.27}$$

calculated up to second-order terms (see further the end of section 9).

In one respect the definition of (2.27) that would follow from section 6 deviates from Gustafson's, for whereas Eq. (2.13) gives $A_\mu^{(\alpha)}$ as $A_\mu^{(\alpha)(1)} + \mathcal{A}_\mu^{(\alpha)}$, all earlier calculations have used the simpler form $A_\mu^{(\alpha)} = A_\mu^{(\alpha)(1)} + \mathcal{A}_\mu$. In his first paper on the α-method, Gustafson[2] proved that the term in (2.27) containing $A_\mu^{(\alpha)(1)}$ is finite, in fact equal to zero, if the retardation is neglected in the calculations. A finite value was also obtained for the corresponding term in the theory of the interaction of a nucleon and a scalar meson field.[3] To avoid divergences arising from the vacuum fluctuations $\mathcal{A}_\mu$ the hypothesis of negative-energy photons had at first to be employed, but as already mentioned Gustafson could later show[4] that this is unnecessary if the Dirac equation (2.25) is also solved by analytic continuation; the value of the resulting term W_{fluct} in (2.27) is, on neglecting the retardation, $-\frac{e^2}{\hbar c} m c^2$ (for $\varepsilon_0 = 1/4\pi$).

In [I] the same calculation (the retardation neglected) has been carried out on the basis of the hole theory for the electrostatic self-energy of the electron, which was shown still to diverge logarithmically, as it does also in the ordinary theory.[5] A generalization was given in [II]: *all logarithmic divergences remain unchanged in the α-method, all other types of divergences being eliminated.* In [II] full account was also taken of the retardation, with the result that the total self-energy of an electron at rest was obtained as zero on the one-electron theory, i. e. in this case $W_{\text{fluct}} = 0$.

The same value resulted from a calculation by Serpe[6], who also pointed out the necessity of using a Hermitian combination in (2.27) in order to avoid imaginary values[7]; he employed the symmetrical arrangement $\frac{1}{2}\{\psi^* \alpha^\mu \psi, A_\mu^{(\alpha)}\}$. The fact that the expectation value of $\operatorname*{cont}_{\alpha=0} \psi^* \alpha^\mu \psi A_\mu^{(\alpha)}$ proves to be imaginary shows that $A_\mu^{(\alpha)}$ does not commute with $\psi^* \alpha^\mu \psi$ or with ψ, not even in the limit $\alpha = 0$. We shall come back to this question in section 9.

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[1] Cf. WEISSKOPF [1], [2].

[2] GUSTAFSON [1].

[3] GUSTAFSON [2], [3].

[4] GUSTAFSON [4], [5]. An erroneous factor has crept into the end result in the first of these papers and is corrected in the second. Cf. also [II], foot-note (*i*).

[5] WEISSKOPF [1] (erratum), [2]. — The notation 'logarithmic divergence' etc. of an expression, supposed to be written as an integral over momentum space, refers to the behaviour of the integral as a function of the upper limit K (if the integration is first extended over a sphere of radius K, where $K \to \infty$).

[6] SERPE [1].

[7] The same imaginary value as Serpe obtains without symmetrization, SERPE [1] Eq. (31), is in [II] contained in the contribution from the term $da_{-1}/d\beta$, § IV A. See further section 9.

In these applications the term W_{fluct} has been worked out using analytic continuation in solving the Dirac equation. This amounts to replacing $\psi^{(1)}$ in the interaction operator by an 'α-solution' $\psi^{(1)(\alpha)}$, which, with the definition used in [II] § II, can be obtained from (2.26) by simply raising the denominators in the bracket terms to the power $1 + \alpha/2$. We shall verify in the next section that the same result for W_{fluct} is obtained from the formalism in section 6, i. e. by introducing $A_\mu^{(\alpha)}$ in place of A_μ into the interaction term in the Dirac equation.

The question of the logarithmic singularities will be taken up again in Chapter IV.

8. Supplementary calculations. In this section two additions will be made to the exposition in [II].[1]

The first concerns some purely formal difficulties in [II]. In evaluating an integral like $\int V^{\alpha+2}(\boldsymbol{x}-\boldsymbol{y})\, e^{i(\boldsymbol{K},\boldsymbol{y})}\,(d\,y)^4$, which is not properly convergent for any value α, we had there to fall back upon the definition of oscillating integrals[2]; that, however, may be considered quite a natural procedure. Then[3]

$$\int V^{\alpha+2}(\boldsymbol{x}-\boldsymbol{y})\, e^{i(\boldsymbol{K},\boldsymbol{y})}\,(d\,y) = \frac{e^{i(\boldsymbol{K},\boldsymbol{x})}}{(\boldsymbol{K},\boldsymbol{K})^{1+\alpha/2}}. \tag{2.28}$$

A more serious difficulty arose in the subsequent integration over momentum space (integration over $\vec{p}$, if $\boldsymbol{K} = \boldsymbol{p} - \boldsymbol{q}$ with $\boldsymbol{q}$ fixed, [II] Eq. (3.4)), since the values of α necessary to make the integral converge for infinitely large momenta might well bring in new divergences, without any counterpart in the case $\alpha = 0$, at other points (for $(\boldsymbol{K},\boldsymbol{K}) = 0$). The device used in [II] for getting round this singularity consisted in a cutting-off procedure for $\alpha > 0$ (see e. g. [II] Eq. (3.5)).

In spite of the seemingly arbitrary character of such cut-offs it should be stressed that the difficulties are only formal and can easily be avoided in the ways indicated in section 4: either by using an initial surface S or by modifying the kernel; the former procedure was also used in GUSTAFSON [1]. The form (1.26) of the kernel ($V_{\alpha,l_0}^{\alpha+2}$), for instance, will cause the denominator $(\boldsymbol{K},\boldsymbol{K})^{1+\alpha/2}$ in (2.28) to be replaced by $[(\boldsymbol{K},\boldsymbol{K}) + \alpha^2/l_0^2]^{1+\alpha/2}$, thereby avoiding the singularity at $(\boldsymbol{K},\boldsymbol{K}) = 0$; cf. section 21 and [II] § IV. Of course the expressions obtained from such more elaborate prescriptions will be rather unwieldy to handle in practical calculations as compared with the simple cut-off used in [II], though the effect is the same; it may therefore be sufficient to establish the formal character of the difficulties and to see how they can be overcome by rigorous methods, in order to get a justification for adopting the cut-off device.

In the second place we need to verify that the use of $A_\mu^{(\alpha)}$ instead of A_μ in (2.25) and (2.27) causes the contribution to the self-energy from the vacuum fluctuations of the field to be convergent, and more exactly to vanish, as in [II].

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[1] An oversight in [II] should also be pointed out: the numerator in the second term of Eq. (4.1) should read $-p_0 + m$ and not $p_0 - m$.

[2] Cf. PAULI [3] § 6 (b) (p. 202).

[3] See [II] Eq. (2.5). When $(\boldsymbol{K},\boldsymbol{K}) < 0$ the branch is determined by $(\boldsymbol{K},\boldsymbol{K})^{-1-\alpha/2} = -|(\boldsymbol{K},\boldsymbol{K})|^{-1-\alpha/2}\, e^{i\alpha\pi/2}$ for $K_0 > 0$, the sign of i being reversed if $K_0 < 0$.

According to (2.23) $\mathcal{A}_\mu$ is of the form

$$\mathcal{A}(\boldsymbol{x}) = \int [U(\boldsymbol{k})\, e^{i(\boldsymbol{k},\boldsymbol{x})} + \text{conj.}]\,(d\,k)^3. \tag{2.29}$$

We have thus to form a function $u^{(\alpha)}$ as in (1.9) with $u = e^{i(\boldsymbol{k},\boldsymbol{x})}$, $(\boldsymbol{k},\boldsymbol{k}) = 0$. If the surface S is chosen to be the plane $y_0 =$ const., the normal derivative occurring in the integral is $d/d\,n = \partial/\partial\,y_0$. Putting $\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{z}$ in (1.9) we get (with $k_0 = |\vec{k}| = k$)

$$u^{(\alpha)}(\boldsymbol{x}) = e^{i(\boldsymbol{k},\boldsymbol{x})} \int \left[- V^{\alpha+2}(\boldsymbol{z})\, e^{-i(\boldsymbol{k},\boldsymbol{z})} \cdot i\,k + \frac{\partial\, V^{\alpha+2}(\boldsymbol{z})}{\partial\, z_0}\, e^{-i(\boldsymbol{k},\,\boldsymbol{z})} \right] (d\,z)^3. \tag{2.30}$$

With $|\vec{z}| = r$, we have for $r < z_0$

$$\begin{aligned} V^{\alpha+2}(\boldsymbol{z}) &= \frac{1}{H(\alpha+2)}\,(z_0^2 - r^2)^{\frac{\alpha}{2}-1}, \\ \frac{\partial\, V^{\alpha+2}(\boldsymbol{z})}{\partial\, z_0} &= \frac{(\alpha-2)}{H(\alpha+2)}\, z_0\,(z_0^2 - r^2)^{\frac{\alpha}{2}-2}. \end{aligned} \tag{2.31}$$

We further write $e^{-i(\boldsymbol{k},\boldsymbol{z})} = e^{i\,k\,z_0}\, e^{-i\,k\,r\,\tau}$, so that $(d\,z)^3 = r^2\, d\,r\, d\,\tau\, d\,\varphi$. The integrations in φ from 0 to $2\,\pi$ and in τ from -1 to $+1$ can then be performed immediately, yielding

$$u^{(\alpha)}(\boldsymbol{x}) = e^{i(\boldsymbol{k},\boldsymbol{x})} \cdot \frac{4\,\pi}{H(\alpha+2)} \int_0^{z_0} \left[(\alpha-2)\,\frac{z_0}{k}\,(z_0^2 - r^2)^{\frac{\alpha}{2}-2} - i\,(z_0^2 - r^2)^{\frac{\alpha}{2}-1} \right] \cdot r\,\sin\,k\,r\,d\,r \cdot e^{i\,k\,z_0} \tag{2.32}$$

which on insertion of the expression for $H(\alpha+2)$ and evaluation reduces to[1]

$$u^{(\alpha)}(\boldsymbol{x}) = e^{i(\boldsymbol{k},\boldsymbol{x})}\, \frac{2^{-\alpha/2}}{\Gamma\left(1+\frac{\alpha}{2}\right)} \sqrt{\frac{\pi}{2}}\, \frac{z_0^{\frac{\alpha+1}{2}}}{k^{\frac{\alpha-1}{2}}}\, [J_{\frac{\alpha-1}{2}}(k\,z_0) - i\,J_{\frac{\alpha+1}{2}}(k\,z_0)]\, e^{i\,k\,z_0}. \tag{2.33}$$

As $\alpha \to 0$ the factor after $e^{i(\boldsymbol{k},\boldsymbol{x})}$ tends to unity independently of z_0, as it should. If this factor is written $F_\alpha(k, z_0)$, $\mathcal{A}^{(\alpha)}$ becomes

$$\mathcal{A}^{(\alpha)}(\boldsymbol{x}) = l_0^{-\alpha} \int [U(\boldsymbol{k})\, F_\alpha(k, z_0)\, e^{i(\boldsymbol{k},\boldsymbol{x})} + \text{conj.}]\,(d\,k). \tag{2.34}$$

Here $z_0 = x_0 - y_0$, and thus F_α varies with $\boldsymbol{x}$ if y_0 is fixed. It will mean simplification, however, if we make a departure from the strict Riesz method by defining $\mathcal{A}^{(\alpha)}(\boldsymbol{x})$ in such a way that the difference $x_0 - y_0$ is kept constant;

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[1] For this and the following formulas involving Bessel functions see e.g. N. Nielsen, *Theorie der Cylinderfunktionen* (Leipzig 1904), or G. N. Watson, *Theory of Bessel Functions* (2nd ed. Cambridge 1944).

that is to say, we let the surface S depend on the field point $\boldsymbol{x}$.[1] The first approximation $\psi^{(1)}$ satisfying (2.14) is then evidently obtained from (2.26) by replacing

$$A_\nu(\boldsymbol{k}) \to l_0^{-\alpha} A_\nu(\boldsymbol{k})\, F_\alpha(k, z_0), \qquad A_\nu^*(\boldsymbol{k}) \to l_0^{-\alpha} A_\nu^*(\boldsymbol{k})\, F_\alpha^*(k, z_0). \tag{2.35}$$

In forming the expectation value $\left\langle \int \psi^{(0)*} \alpha^\mu \psi^{(1)} A_\mu^{(\alpha)} (d\,x)^3 \right\rangle$ for an electron at rest — with the supplementary condition $\frac{\partial A_\mu^{(\alpha)}}{\partial x_\mu} | \rangle = 0$ taken into consideration — one is now led to expressions of the form

$$\int_0^\infty F_\alpha(k, z_0)\, F_\alpha^*(k, z_0)\, k^n\, d\,k \tag{2.36}$$

(here in fact with $n = 1$). Since

$$F_\alpha(k, z_0)\, F_\alpha^*(k, z_0) = \frac{2^{-\alpha}}{\Gamma\left(1 + \frac{\alpha}{2}\right)^2} \frac{\pi}{2} z_0^{\alpha+1} k^{1-\alpha} \left[J_{\frac{\alpha-1}{2}}(k\,z_0)^2 + J_{\frac{\alpha+1}{2}}(k\,z_0)^2\right], \tag{2.37}$$

we have then to consider integrals of the type

$$\int_0^\infty k^q J_p(k\,z_0)^2\, d\,k = \frac{\cos q\,\pi/2}{2\,\pi^{3/2} z_0^{1+q}} \frac{\Gamma\left(p + \frac{1+q}{2}\right) \Gamma\left(\frac{1+q}{2}\right) \Gamma\left(-\frac{q}{2}\right)}{\Gamma\left(p + \frac{1-q}{2}\right)} \tag{2.38}$$

for $q = n + 1 - \alpha$, $p = \frac{1}{2}(\alpha \pm 1)$ (convergence for $n > -1$, $\alpha > n + 1$). The sum for $p = \frac{1}{2}(\alpha - 1)$ and $p = \frac{1}{2}(\alpha + 1)$ is easily simplified by using the elementary properties of Γ-functions; thus

$$\int_0^\infty F_\alpha(k, z_0)\, F_\alpha^*(k, z_0)\, k^n\, d\,k =$$

$$= z_0^{2\alpha-n-1} \frac{\sin (n - \alpha)\,\pi/2}{2^{1+\alpha}\,\pi^{1/2}\,\Gamma\left(1 + \frac{\alpha}{2}\right)^2} \frac{\Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{n+2-\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2} - \frac{n+1}{2}\right)}{(n + 1 - 2\,\alpha)\,\Gamma\left(\alpha - \frac{n+1}{2}\right)} \cdot \alpha, \tag{2.39}$$

so that

$$\operatorname*{cont}_{\alpha=0} \int_0^\infty F_\alpha(k, z_0)\, F_\alpha^*(k, z_0)\, k^n\, d\,k = 0 \qquad (\text{for } n > -1) \tag{2.40}$$

irrespective of the value of z_0.

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[1] It might seem more satisfactory to let S be an hyperboloid surface $R = \text{const.}$, the expression for $A^{(\alpha)}$ then being relativistic; but the very relativistic character would destroy the possibility of eliminating divergences with the help of $A^{(\alpha)}$, since the dependence of the surface integral on $\boldsymbol{k}$ could then only involve the invariant $(\boldsymbol{k}, \boldsymbol{k}) = 0$; cf. Chap. III and section 28. It is therefore rather natural to treat the vacuum field by itself and to remove the divergences by using analytic continuation in solving the Dirac equation, as proposed by Gustafson. — The end result (2.41) would be the same for a fixed surface S.

In particular we get the desired result

$$W_{\text{fluct}} = 0 \tag{2.41}$$

also from this calculation.

9. **Consistency questions.** We have postponed till now a discussion of the problem raised in section 6 about the consistency of the definitions used for energy-momentum quantities; the requirement was that Eq. (2.16) should be fulfilled. We shall here verify, taking the simplest example of an electron at rest (no external field), that the expectation value of the left-hand side in (2.16) does indeed vanish in the lowest significant approximation.

According to Eq. (2.17) and the discussion following it, it is then necessary to make a comparison between

$$\varepsilon_0 \left\{ \Box A^{\nu(\alpha)}, \frac{\partial A_\nu^{(\alpha)}}{\partial x^\mu} \right\} \quad \text{and} \quad i\,e \left\{ \psi^\dagger \gamma^\nu \psi, \frac{\partial A_\nu^{(\alpha)}}{\partial x^\mu} \right\}.$$

Using perturbation theory up to second-order terms, and dropping terms whose expectation values vanish, we have

$$\varepsilon_0 \left\{ \Box A^{\nu(\alpha)}, \frac{\partial A_\nu^{(\alpha)}}{\partial x^\mu} \right\} = \varepsilon_0 \left\{ \Box A^{\nu(\alpha)(1)}, \frac{\partial A_\nu^{(\alpha)(1)}}{\partial x^\mu} \right\} + \varepsilon_0 \left\{ \Box A^{\nu(\alpha)(2)}, \frac{\partial A_\nu^{(\alpha)}}{\partial x^\mu} \right\}, \tag{2.42}$$

$$i\,e \left\{ \psi^\dagger \gamma^\nu \psi, \frac{\partial A_\nu^{(\alpha)}}{\partial x^\mu} \right\} =$$

$$= i\,e \left\{ \psi^{\dagger(0)} \gamma^\nu \psi^{(0)}, \frac{\partial A_\nu^{(\alpha)(1)}}{\partial x^\mu} \right\} + i\,e \left\{ \psi^{\dagger(0)} \gamma^\nu \psi^{(1)} + \psi^{\dagger(1)} \gamma^\nu \psi^{(0)}, \frac{\partial A_\nu^{(\alpha)}}{\partial x^\mu} \right\}. \tag{2.43}$$

$A_\nu^{(\alpha)(1)}$ is given by (2.22); for convenience we write down the expression again:

$$A_\nu^{(\alpha)(1)}(\boldsymbol{x}) = -\frac{e}{\varepsilon_0} \frac{l_0^{-\alpha}}{(2\pi)^3} \sum_n \sum_m \int\!\!\int a_m'^{*} a_n (u_m'^{*} \alpha_\nu u_n) \cdot$$

$$\cdot \frac{e^{i(\boldsymbol{p}_n - \boldsymbol{p}_m', \boldsymbol{x})}}{(\boldsymbol{p}_n - \boldsymbol{p}_m', \boldsymbol{p}_n - \boldsymbol{p}_m')^{1+\alpha/2}} (d\,p)(d\,p'). \tag{2.44}$$

Therefore

$$\varepsilon_0 \Box A_\nu^{(\alpha)(1)} = e \frac{l_0^{-\alpha}}{(2\pi)^3} \sum_t \sum_s \int\!\!\int a_s'''^{*} a_t'' (u_s'''^{*} \alpha_\nu u_t'') \cdot$$

$$\cdot \frac{e^{i(\boldsymbol{p}_t'' - \boldsymbol{p}_s''', \boldsymbol{x})}}{(\boldsymbol{p}_t'' - \boldsymbol{p}_s''', \boldsymbol{p}_t'' - \boldsymbol{p}_s''')^{\alpha/2}} (d\,p'')(d\,p'''), \tag{2.45}$$

from which $i\,e\,\psi^{\dagger(0)} \gamma_\nu \psi^{(0)}$ is obtained by putting $\alpha = 0$. Now the expectation value of $a_s'''^{*} a_t'' a_m'^{*} a_n$ in a state representing one single electron (of momentum $\vec{q}$ and spin coordinate r) vanishes unless $\vec{p}''' = \vec{p}$, $\vec{p}'' = \vec{p}'$, and $s = n$, $t = m$; in fact

$$\langle a_s^*(\vec{p}\,''')\, a_t(\vec{p}\,'')\, a_m^*(\vec{p}\,')\, a_n(\vec{p}\,)\rangle =$$

$$= \delta_{nr}\,\delta_{sr}\,\delta_{tm}\,\delta(\vec{p}-\vec{q}\,)^{1/2}\,\delta(\vec{p}\,'''-\vec{q}\,)^{1/2}\,\delta(\vec{p}\,''-\vec{p}\,')\,. \qquad (2.46)$$

The expectation value of the first term on the right in (2.42) is therefore seen to differ from the corresponding term in (2.43) only by 2α taking the place of α. Strictly speaking this is not quite true, for when the scalar product in the denominators of (2.44) and (2.45) is negative, the factors $e^{i\alpha\pi/2}$ and $e^{-i\alpha\pi/2}$ will cancel in (2.42) (according to note 3 on p. 389). However, owing to the symmetrization in (2.43), it will not make any difference in the results if we there replace $e^{\pm i\alpha\pi/2} \to 1$, and then the agreement of the first terms in (2.42) and (2.43) is complete but for the interchange $2\alpha \longleftrightarrow \alpha$, which has no influence on the continuation to $\alpha = 0$.

The comparison between the second terms in (2.42) and (2.43) is not so straightforward, and we shall not here enter into details, contenting ourselves with the remark that no real difficulty arises if a boundary surface S is used in the calculations.[1]

It remains to investigate the second term in (2.17), which may be written

$$i\,e\,\psi^\dagger\gamma^\nu\frac{\partial A_\nu^{(\alpha)}}{\partial x^\mu}\,\psi = \frac{e}{2}\left\{\psi^*\,\alpha^\nu\psi,\ \frac{\partial A_\nu^{(\alpha)}}{\partial x^\mu}\right\} + \frac{e}{2}\,\psi^*\,\alpha^\nu\left[\frac{\partial A_\nu^{(\alpha)}}{\partial x^\mu},\ \psi\right] +$$

$$+\frac{e}{2}\left[\psi^*,\ \frac{\partial A_\nu^{(\alpha)}}{\partial x^\mu}\right]\alpha^\nu\,\psi\,. \qquad (2.47)$$

According to (2.17) and the considerations above, we have then to require that the expectation value of the last two terms in (2.47) (or their integrals) shall vanish for $\alpha \to 0$. In the ordinary theory that follows immediately, since A_ν and its derivatives of the first order commute with ψ at the same point; here, however, a special investigation is needed, for $A_\nu^{(\alpha)}$ and ψ certainly do not commute for $\alpha > 0$ at any rate. Indeed it was pointed out in section 7 that they cannot commute even in the limit $\alpha = 0$.

To see this explicitly we turn to the commutator

$$[\psi(\boldsymbol{y}),\ A_\nu^{(\alpha)}(\boldsymbol{x})] \text{ with } y_0 = x_0\,. \qquad (2.48)$$

Using the expressions for $\psi^{(0)}$ and $A_\nu^{(\alpha)(1)}$, Eqs. (2.20) and (2.44), and further the commutation relation (2.21), we can first form $[\psi^{(0)}(\boldsymbol{y}),\ A_\nu^{(\alpha)(1)}(\boldsymbol{x})]$, which after some calculations can be reduced to

$$[\psi^{(0)}(\boldsymbol{y}),\ A_\nu^{(\alpha)(1)}(\boldsymbol{x})] = -\frac{e}{\varepsilon_0}\,\frac{l_0^{-\alpha}}{(2\pi)^{9/2}}\sum_n\int(d\,p)\,a_n\,e^{i(\boldsymbol{p}_n,\,\boldsymbol{x})}\sum_r\int(d\,p')\,u_r'\,u_r'^*\,\alpha_\nu\,u_n\,\cdot$$

$$\cdot\frac{e^{i(\boldsymbol{p}_r',\,\boldsymbol{y}-\boldsymbol{x})}}{(\boldsymbol{p}_n-\boldsymbol{p}_r',\ \boldsymbol{p}_n-\boldsymbol{p}_r')^{1+\alpha/2}} = \qquad (2.49)$$

---

[1] Otherwise complications are caused by the fact that $\overset{(2)}{A_\mu}$ satisfies an equation of the type $\Box\, u = \exp i\,(\boldsymbol{k},\,\boldsymbol{x})$ with $(\boldsymbol{k},\,\boldsymbol{k}) = 0$.

$$= -\frac{e}{\varepsilon_0}\frac{2^{-2-\alpha/2}}{(2\pi)^{9/2}} l_0^{-\alpha} \sum_n \int (d p)\, a_n e^{i(p_n, y)} \cdot \int \frac{(d k)}{[E(\vec{p})^2 k^2 - (\vec{p}\,\vec{k})^2]^{1+\alpha/2}} e^{i\vec{k}(\vec{y}-\vec{x})} \cdot$$
$$\cdot E(\vec{p}+\vec{k})^{-1}[(\vec{k}\,\vec{\alpha} + \vec{p}\,\vec{\alpha} + m\beta + E(\vec{p}+\vec{k}))\, B_n^+(\vec{p},\vec{k})^{1+\alpha/2} -$$
$$-(\vec{k}\,\vec{\alpha} + \vec{p}\,\vec{\alpha} + m\beta - E(\vec{p}+\vec{k}))\, B_n^-(\vec{p},\vec{k})^{1+\alpha/2}]\, \alpha_\nu u_n(\vec{p})$$

(cut-off for small values of k when $\alpha > 0$; see section 8) with

$$E(\vec{p}) = |p_0| = (p^2 + m^2)^{1/2} \tag{2.50}$$

(i. e. $p_{n0} = E(\vec{p})$ or $-E(\vec{p})$ according as $n = 1, 2$ or $3, 4$) and

$$B_n^{\pm}(\vec{p},\vec{k}) = E(\vec{p})^2 + \vec{p}\,\vec{k} \pm E(\vec{p}+\vec{k})\, p_{n0}. \tag{2.51}$$

For $p_{n0} > 0$ we have $(B_n^-)^{1+\alpha/2} = -|B_n^-|^{1+\alpha/2} e^{+i\alpha\pi/2}$, while $(B_n^+)^{1+\alpha/2} = -|B_n^+|^{1+\alpha/2} e^{-i\alpha\pi/2}$ when $p_{n0} < 0$ (see note 3 on p. 389). In (2.49) $\vec{k}$ has been written for $\vec{p}\,' - \vec{p}$, which puts the expression in a form that can easily be compared with $[\psi^{(1)}(\boldsymbol{y}), A_\nu^{(\alpha)}(\boldsymbol{x})]$:

$$[\psi^{(1)}(\boldsymbol{y}), A_\nu^{(\alpha)}(\boldsymbol{x})] = \frac{e}{\varepsilon_0}\frac{2^{-2}}{(2\pi)^{9/2}} l_0^{-2\alpha} \sum_n \int (d p)\, a_n e^{i(p_n, y)} \cdot$$
$$\cdot \int \frac{(d k)}{E(\vec{p})^2 k^2 - (\vec{p}\,\vec{k})^2} e^{i\vec{k}(\vec{y}-\vec{x})} \cdot (F_\alpha F_\alpha^*) \cdot \tag{2.52}$$
$$\cdot [(\vec{k}\,\vec{\alpha} + \vec{p}\,\vec{\alpha} + m\beta + p_{n0}) \cdot 2 p_{n0} + 2\,\vec{p}\,\vec{k}]\, \alpha_\nu u_n(\vec{p})$$

(cf. (2.34), (2.35)). The sum of (2.49) and (2.52) gives the first-order approximation of the commutator (2.48); if α is set equal to nought in the integrands the sum vanishes.

However, when the calculation is performed by analytic continuation, the result is zero only for $\vec{y} \neq \vec{x}$; if the fields are considered at the same point we get finite contributions from those terms in the integrand of (2.49) which behave as $\alpha k^{-1-\alpha/2}$ for large values of k. Thus

$$[\psi(\boldsymbol{x}), A_\nu(\boldsymbol{x})] = -\frac{e}{\varepsilon_0}\frac{1}{4(2\pi)^{9/2}} \sum_n \int (d p)\, a_n(\vec{p})\, e^{i(p_n, x)} f_n(\vec{p})\, \alpha_\nu u_n(\vec{p}) \tag{2.53}$$

with a non-vanishing matrix function $f_n(\vec{p})$. If, for example, we take out the term that is of importance for an electron at rest ($\vec{p} = 0$, $p_{n0} = +m$), we find on writing $(d k) = k^2\, d k\, d\omega$ and expanding in powers of $1/k$ that the k-integrand in (2.49), for large values of k, can be expressed as

$$\frac{1}{m}(\vec{k}\,\vec{\alpha} + m\beta)\left[2 + \frac{i\alpha\pi}{2} + \frac{i\alpha\pi}{2}\frac{m}{k} + \alpha\frac{m^2}{k^2} + \alpha\, O(1/k^3)\right] k^{-\alpha/2} +$$
$$+ \left[-\frac{i\alpha\pi}{2}\frac{k}{m} + \left(2 + \alpha + \frac{i\alpha\pi}{2}\right) - \frac{i\alpha\pi}{4}\frac{m}{k} + \alpha\, O(1/k^2)\right] k^{-\alpha/2} + O(\alpha^2), \tag{2.54}$$

which gives

$$f_n(0) = -2\pi^2 i(1+2\beta)m \qquad \text{for } n = 1, 2. \tag{2.55}$$

This follows by integration of (2.54): $\oint d\omega \int\limits_K^\infty dk \ldots$, the terms in (2.54) not proportional to α being cancelled by corresponding terms in (2.52), and the contributions from $\int\limits_0^K dk \ldots$ by the same integral in (2.52), when $\alpha \to 0$.

In the same way we may write

$$\left[\psi(\boldsymbol{x}),\ \frac{\partial A_\nu(\boldsymbol{x})}{\partial x^\mu}\right] = -\frac{e}{\varepsilon_0}\frac{1}{4(2\pi)^{9/2}}\sum_n \int (dp)\, a_n(\vec{p})\, e^{i(p_n, x)} f_{n\mu}(\vec{p})\, \alpha_\nu\, u_n(\vec{p}), \tag{2.56}$$

with another function $f_{n\mu}(\vec{p})$. By a similar calculation to that outlined above one finds that $f_{n\mu}(0)$ vanishes for $\mu = 0$ and is proportional to α_k for $\mu = k$ $(= 1, 2, 3)$; therefore the expectation value of the last two terms in (2.47) is seen to vanish (because of $u_n^*(0)\,\alpha^\nu \alpha_k \alpha_\nu u_n(0) = 0$). That is the result we wanted to establish.

The imaginary form of $f_n(0)$ shows that the self-energy operator (in the e^2 approximation) can equally well be defined as half the interaction term in (2.15),

$$W = \operatorname*{cont}_{\alpha=0} \frac{e}{2}\int \psi^* \alpha^\nu A_\nu^{(\alpha)} \psi\, (dx)^3, \tag{2.57}$$

or as

$$W = \operatorname*{cont}_{\alpha=0} \frac{e}{4}\int \{\psi^* \alpha^\nu \psi,\ A_\nu^{(\alpha)}\}\, (dx)^3; \tag{2.58}$$

the expectation values are the same.[1] In the last expression $e\,\psi^* \alpha^\nu \psi$ may also be replaced by $\varepsilon_0 \Box A^{\nu(\alpha)}$ (as in the beginning of this section). The equality of these expressions gives a justification for using the simple form (2.57) or (2.58) instead of the more complicated expression that would be obtained by forming the expectation value directly of H in (2.15).

That the form (2.27) is not correct then follows from (2.53) and (2.55). We may check (2.55) with Serpe's imaginary value for (2.27). Thus for $\mu = 0$

$$-\operatorname*{cont}_{\alpha=0}\frac{e}{2}\int \psi^* \psi A_0^{(\alpha)}\, (dx) = -\operatorname*{cont}_{\alpha=0}\frac{e}{2}\int \psi^* A_0^{(\alpha)} \psi\, (dx) - \\ -\frac{e}{2}\int \psi^* [\psi, A_0]\, (dx). \tag{2.59}$$

The expectation value of the last term for a state with $\vec{p} = 0$, spin variable r, is

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[1] Note that if (2.57) is used the contribution from the term containing $A_\nu^{(\alpha)(1)}$ is automatically zero, owing to the changed order of the factors; only the vacuum fluctuations $\mathcal{A}_\mu$ will then come in. This indicates the formal nature of the decomposition into W_{st}, W_{sp}, and W_{fluct}, used in [II].

$$-\frac{e}{2}\left\langle \int \psi^{*}\,[\psi, A_0]\,(d\,x)\right\rangle = \frac{e^2}{8\,\varepsilon_0}\,\frac{1}{(2\,\pi)^6}\sum_m\sum_n \int\!\!\int (d\,p)\,(d\,p')\,\langle a_m^{*}(\vec{p}\,')\,a_n(\vec{p})\rangle\,\cdot$$

$$\cdot\,(u_m^{*}(\vec{p}\,')\,f_n(\vec{p})\,u_n(\vec{p}))\int e^{i(p_n - p'_m,\, x)}\,(d\,x) = \tag{2.60}$$

$$= \frac{e^2}{8\,\varepsilon_0}\,\frac{1}{(2\,\pi)^3}\,u_r^{*}(0)\,f_r(0)\,u_r(0) = -\frac{e^2}{8\,\varepsilon_0}\,\frac{2\,\pi^2\,i}{(2\,\pi)^3}\cdot 3\,m = -\frac{3}{8}\,\frac{e^2\,m}{4\,\pi\,\varepsilon_0}\,i\,,$$

which is the same expression as in SERPE [1] Eq. (31) (for $\varepsilon_0 = 1/4\,\pi$). (In (2.60) we have used

$$\langle a_m^{*}(\vec{p}\,')\,a_n(\vec{p})\rangle = \delta_{m\,r}\,\delta_{n\,r}\,\delta(\vec{p}\,')^{1/2}\,\delta(\vec{p})^{1/2},\quad \frac{1}{(2\,\pi)^3}\int e i(\vec{p}-\vec{p}\,')\vec{x}\,(d\,x) = \delta(\vec{p}-\vec{p}\,').)$$

10. Remark. Even if the non-commutability established in section 9 does not destroy the consistency of the scheme (in the case we have investigated), it may nevertheless be regarded as an undesirable feature of the theory; for instance, a reversal of the order of the factors in Eq. (2.14) will change the equation. Now the fact that quantities like $A_\nu(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$ do not commute at the same point when analytic continuation is used as above is a consequence of the formalism not being Hamiltonian for general α's. If we could give it a Hamiltonian form, such complications would be automatically eliminated. That task will be undertaken in the next chapter, where we shall see, however, that the relativistic character of the theory cannot then be maintained for general α's.

CHAPTER III

A Hamiltonian formalism

11. Preliminaries. In the last section the desirability of making the α-continuation method into a canonical scheme was pointed out. Such a scheme has been proposed by Pauli[1], starting from a many-time formulation of quantum electrodynamics[2] in which a configuration space is used for the electrons, the state vector being a function of the position vectors and spin variables of all the particles; the formalism can be obtained from that in section 5 by a canonical transformation. For every electron one then has[3]

$$\left[\frac{\hbar}{i}\,\alpha_\mu\,\frac{\partial}{\partial z_\mu} + m\,c\,\beta + \frac{e}{c}\,A_\mu(\boldsymbol{z})\,\alpha^\mu\right]|\,\boldsymbol{z}\,\rangle = 0\,, \tag{3.1}$$

the electromagnetic potential vector satisfying

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[1] See [II] § V.

[2] DIRAC-FOCK-PODOLSKY [1], see also WENTZEL [2] § 18.

[3] The notation for the state vector, Dirac's 'ket vector', follows the 3rd edition of Dirac's book (1947), DIRAC [5] §§ 20, 79.

$$\Box A_\mu(\boldsymbol{x}) = 0 . \tag{3.2}$$

As a generalization of (2.6) one has[1]

$$[A_\mu(\boldsymbol{x}), A_\nu(\boldsymbol{y})] = -\frac{i\hbar c}{\varepsilon_0} g_{\mu\nu} D(\boldsymbol{x}-\boldsymbol{y}), \tag{3.3}$$

and the supplementary condition (2. 9) is replaced by

$$\left[\frac{\partial A_\mu(\boldsymbol{x})}{\partial x_\mu} - \sum \frac{e}{\varepsilon_0} D(\boldsymbol{x}-\boldsymbol{z})\right] |\boldsymbol{z}\rangle = 0, \tag{3.4}$$

the summation being extended over all the electrons ($\boldsymbol{z} = \boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)} \ldots$).

Pauli's proposal amounts to modifying the commutation relation (3.3) in a certain way by letting the commutator be a function of a parameter α, afterwards making an analytic continuation to $\alpha = 0$. Since the connection with the Riesz method is not immediately apparent, we intend here to approach the problem from the other side, by going back to the classical theory and modifying the Riesz kernel, after the manner of section 4, in such a way as to make it suitable for a Hamiltonian formalism. It will turn out that in this way one is again led to Pauli's formulation, in a somewhat generalized form.

Classical theory

12. **Hamiltonian form of the equations of motion.** We begin with a short exposition of a formalism due to Dirac[2] intended to put the equations of motion into Hamiltonian form. With the position four-vector of the electron denoted by $\boldsymbol{z} = \boldsymbol{z}(s)$ as before, a relativistic Hamiltonian for the electron is introduced:

$$G(s) = \frac{1}{2mc}\left[\left(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}(\boldsymbol{z}), \boldsymbol{p} + \frac{e}{c}\boldsymbol{A}(\boldsymbol{z})\right) + m^2 c^2\right]. \tag{3.5}$$

A_μ is in our case to be a generalized electromagnetic field potential depending on a parameter α. The equation of motion of a quantity F that depends on s is to be a consequence of the relation

$$\frac{dF}{ds} = [F, G], \tag{3.6}$$

where the right-hand side denotes the Poisson bracket of F and G. In the presence of many electrons, variables $s_1, s_2 \ldots$ are introduced and in the same manner $G_1, G_2 \ldots$; an equation of type (3.6) will then hold for the dependence of F on each s_n.

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[1] $D(\boldsymbol{x})$ is the invariant D or Δ function of Jordan and Pauli. Different conventions are used in the literature with respect to the sign and the normalization; here we adopt the definition $D(\boldsymbol{x}) = \frac{1}{4\pi r}[\delta(x_0 - r) - \delta(x_0 + r)]$ (with $r = |\vec{x}|$).

[2] Dirac [2], [4]. The formalism is also to be found in Dirac [5] § 78.

Values of Poisson brackets:

$$[z_\mu, z_\nu] = [p_\mu, p_\nu] = 0, \quad [p_\mu, z_\nu] = -g_{\mu\nu}, \tag{3.7}$$

$$[z_\mu, A_\nu(\boldsymbol{x})] = [p_\mu, A_\nu(\boldsymbol{x})] = 0. \tag{3.8}$$

The Poisson bracket of two quantities referring to different electrons vanishes.

Putting F in (3.6) successively equal to z_μ and p_μ one obtains

$$m c \dot{z}_\mu = p_\mu + \frac{e}{c} A_\mu(\boldsymbol{z}) \tag{3.9}$$

and (using (3.9))

$$m c \ddot{z}_\mu = -\frac{e}{c}\left[\frac{\partial A_\nu(\boldsymbol{z})}{\partial z^\mu} - \frac{\partial A_\mu(\boldsymbol{z})}{\partial z^\nu}\right] \dot{z}^\nu = -\frac{e}{c} F_{\mu\nu}(\boldsymbol{z}) \dot{z}^\nu. \tag{3.10}$$

It follows from (3.9) that $G(s)$ has the value zero; as a general compatibility condition one has to require, in a many-electron problem,

$$[G_m, G_n] = 0. \tag{3.11}$$

With F equal to $A_\mu(\boldsymbol{x})$, (3.6) gives

$$\frac{d A_\mu(\boldsymbol{x})}{d s} = \frac{e}{m c^2}\left(p^\nu + \frac{e}{c} A^\nu(\boldsymbol{z})\right)[A_\mu(\boldsymbol{x}), A_\nu(\boldsymbol{z})] = \frac{e}{c}[A_\mu(\boldsymbol{x}), A_\nu(\boldsymbol{z})]\, \dot{z}^\nu, \tag{3.12}$$

that is,

$$A_\mu(\boldsymbol{x}) = \frac{e}{c} \int\limits_{-\infty}^{s} [A_\mu(\boldsymbol{x}), A_\nu(\boldsymbol{z})]\, u^\nu\, d s + \mathcal{A}_\mu(\boldsymbol{x}) \tag{3.13}$$

with $\mathcal{A}_\mu$ independent of s; this term gives the effect of other electrons and of an external field.

The integral in (3.13) is determined if we have defined the Poisson brackets for the field. In this definition we deviate from Dirac, since we want the new field $A_\mu(\boldsymbol{x})$ to be as closely equal to the Riesz field (1.11) or (1.30) as possible. We write formally

$$[A_\mu(\boldsymbol{x}), A_\nu(\boldsymbol{y})] = -\frac{c}{\varepsilon_0} g_{\mu\nu} D^\alpha(\boldsymbol{x} - \boldsymbol{y}), \tag{3.14}$$

so that, if $\mathcal{A}_\mu = 0$,

$$A_\mu(\boldsymbol{x}) \equiv A_\mu(\boldsymbol{x}, s) = -\frac{e}{\varepsilon_0} \int\limits_{z_\mu(-\infty)}^{z_\mu(s)} D^\alpha(\boldsymbol{x} - \boldsymbol{z})\, d z_\mu. \tag{3.15}$$

Comparing with (1.30), we may tentatively put $D^\alpha(\boldsymbol{x})$ equal to $\bar{V}^\alpha(\boldsymbol{x}) - \bar{V}^\alpha(-\boldsymbol{x})$; that such a combination must be used follows from the fact that D^α, according to (3.14), has evidently to be an odd function of its argument:

$$D^{\alpha}(\boldsymbol{x}) = -D^{\alpha}(-\boldsymbol{x}). \tag{3.16}$$

The field $A_{\mu}(\boldsymbol{x}, s)$ would then bear the same relation to $A_{\mu}^{(\alpha)}(\boldsymbol{x})$ as the Wentzel field[1] to the ordinary retarded field. However, we have to satisfy ourselves whether such a choice of D^{α} would not give rise to contradictions.

We may first remark that, if Eqs. (3.5)—(3.8) and (3.14) are regarded as the fundamental equations of the theory, one has still no field equations for A_{μ}, since A_{μ} in (3.13) can be any function of $\boldsymbol{x}$. To complete the formalism it is necessary, just as in Dirac's treatment, to add field equations (e. g. $\Box A_{\mu} = 0$) and some further condition on the solution corresponding to the divergence relation (1.35).

From (3.15) we get, as a generalization of (1.35),

$$\frac{\partial A_{\mu}(\boldsymbol{x}, s)}{\partial x_{\mu}} = \frac{e}{\varepsilon_0} D^{\alpha}(\boldsymbol{x} - \boldsymbol{z}). \tag{3.17}$$

In the presence of many electrons the right member is replaced by a sum over the particles, so that

$$\mathfrak{R}(\boldsymbol{x}, s) \equiv \frac{\partial A_{\mu}(\boldsymbol{x}, s)}{\partial x_{\mu}} - \sum \frac{e}{\varepsilon_0} D^{\alpha}(\boldsymbol{x} - \boldsymbol{z}) = 0. \tag{3.18}$$

Since $\mathfrak{R}$ has not vanishing Poisson brackets with all other quantities, (3.18) cannot hold as an identity in the Hamiltonian theory. But

$$[\mathfrak{R}(\boldsymbol{x}, s), G(s)] = 0, \tag{3.19}$$

and therefore $\mathfrak{R}$ is a constant of the motion, which may be given the value zero.

According to Poisson's theorem, the Poisson bracket of $\mathfrak{R}(\boldsymbol{x}, s)$ and $\mathfrak{R}(\boldsymbol{y}, s)$ is then also a constant of the motion. Explicitly we get

$$[\mathfrak{R}(\boldsymbol{x}, s), \mathfrak{R}(\boldsymbol{y}, s)] = \frac{\partial}{\partial x_{\mu}} \frac{\partial}{\partial y_{\nu}} [A_{\mu}(\boldsymbol{x}), A_{\nu}(\boldsymbol{y})] = \frac{c}{\varepsilon_0} \Box D^{\alpha}(\boldsymbol{x} - \boldsymbol{y}), \tag{3.20}$$

which is in fact independent of s. But evidently this new constant of the motion cannot be chosen equal to zero unless $\Box D^{\alpha} = 0$; and if we want to carry over the formalism into the quantum theory, where a commutator takes the place of the Poisson bracket (cf. Eq. (3.56)), Eq. (3.20) will then lead to a contradiction for other values of $\Box D^{\alpha}$. For the supplementary condition on $\mathfrak{R}$ that it should be zero will there correspond to the state vector being chosen in such a way as to be annihilated by the operator $\mathfrak{R}(\boldsymbol{x})$ for all points $\boldsymbol{x}$, and therefore by the commutator of $\mathfrak{R}(\boldsymbol{x})$ and $\mathfrak{R}(\boldsymbol{y})$, too.

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[1] WENTZEL [1] I. See also foot-note 2, p. 397.

We thus require that

$$\Box\, D^{\alpha}(\boldsymbol{x}) = 0. \tag{3.21}$$

It is then possible to choose

$$\Box\, A_{\mu}(\boldsymbol{x}, s) = 0 \tag{3.22}$$

as the field equations for A_{μ}. This is satisfactory because Eq. (3.22) should in any case hold for our generalized field when $\alpha = 0$, the current term in Eq. (1.3) being taken account of by the dependence upon s.

13. Determination of D^{α}. To fix the form of D^{α} we have to make use of Eq. (3.21) and of the further condition that it should behave somewhat in the fashion of the generalized Riesz kernel $\bar{V}^{\alpha}(\boldsymbol{x}) - \bar{V}^{\alpha}(-\boldsymbol{x})$. It will be convenient to choose another normalization of the parameter, so that

$$D^{\alpha}(\boldsymbol{x}) \sim \bar{V}^{2\alpha}(\boldsymbol{x}) - \bar{V}^{2\alpha}(-\boldsymbol{x}), \tag{3.23}$$

where, from (1.25), $\bar{V}^{2\alpha}$ is of the form

$$\bar{V}^{2\alpha}(\boldsymbol{x}) = \frac{\alpha}{2\pi} R(\boldsymbol{x})^{\alpha-1} w_{2\alpha}(\boldsymbol{x}) \tag{3.24}$$

inside the direct light-cone from the origin ($R > 0, x_0 > 0$, R being the square of the Lorentz distance), vanishing outside.

It is easy to see that Eq. (3.21) cannot be fulfilled with $w_{2\alpha}$ depending only on the invariant R, and so D^{α} cannot be relativistically invariant for general α's. Moreover, Eq. (3.21) shows that if D^{α} is written as a four-dimensional Fourier integral, it will be of the form

$$D^{\alpha}(\boldsymbol{x}) = \frac{1}{(2\pi)^3} \frac{1}{i} \int e^{i(\boldsymbol{k}, \boldsymbol{x})} g_{\alpha}(\boldsymbol{k})\, \varepsilon(\boldsymbol{k})\, \delta(\boldsymbol{k}, \boldsymbol{k})\, (dk)^4, \tag{3.25}$$

i.e. it gets contributions only for $(\boldsymbol{k}, \boldsymbol{k}) = 0$. The factor in front has been chosen in anticipation of later results[1], and the factor $\varepsilon(\boldsymbol{k}) = \operatorname{sgn} k_0$ has been introduced in order that the simple relation

$$g_{\alpha}(\boldsymbol{k}) = g_{\alpha}(-\boldsymbol{k}) \tag{3.26}$$

may ensure the validity of Eq. (3.16).

From (3.25) and (3.26):

$$\begin{aligned} D^{\alpha}(\boldsymbol{x}) &= \frac{1}{(2\pi)^3} \frac{1}{2i} \int (dk)^3 \int_{-\infty}^{\infty} dk_0\, e^{i(\boldsymbol{k}, \boldsymbol{x})} g_{\alpha}(\boldsymbol{k}) \frac{\delta(k_0 - k) + \delta(k_0 + k)}{k_0} = \\ &= \frac{1}{(2\pi)^3} \int \frac{e^{i(\boldsymbol{k}, \boldsymbol{x})} - e^{-i(\boldsymbol{k}, \boldsymbol{x})}}{2i} g_{\alpha}(\boldsymbol{k}) \frac{(dk)^3}{k_0}. \end{aligned} \tag{3.27}$$

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[1] For $g_{\alpha} = 1$ (3.25) gives the ordinary D function, note 1 p. 397.

Here $k = |\vec{k}|$, and in the last member k_0 stands for the positive value $k_0 = k$.

Obviously $g_\alpha(\boldsymbol{k})$ cannot be a function of the invariant $(\boldsymbol{k}, \boldsymbol{k}) = 0$; the simplest choice is then to take it as depending only on the scalar product $(\boldsymbol{l}, \boldsymbol{k})$, $\boldsymbol{l}$ being a fixed time-like vector of length l_0 say: $g_\alpha(\boldsymbol{k}) = g_\alpha(\boldsymbol{l}, \boldsymbol{k})$.

To simplify (3.27) we employ a special Lorentz system in which the vector $\boldsymbol{l}$ coincides with the time axis, so that $(\boldsymbol{l}, \boldsymbol{k}) = -l_0 k_0 = -l_0 k$. Then, with $|\vec{x}| = r$, $\cos(\vec{k}, \vec{x}) = \tau$,

$$D^\alpha(\boldsymbol{x}) = \frac{1}{(2\pi)^3} \int e^{i\vec{k}\vec{x}} \sin(k_0 x_0)\, g_\alpha(l_0 k) \frac{(dk)^3}{k} =$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty dk \sin(k x_0)\, k\, g_\alpha(l_0 k) \int_{-1}^{1} e^{ikr\tau} d\tau = \frac{1}{2\pi^2 r} \int_0^\infty g_\alpha(l_0 k) \sin k x_0 \sin k r\, dk \tag{3.28}$$

or

$$D^\alpha(\boldsymbol{x}) = \frac{1}{4\pi^2 r} \int_0^\infty g_\alpha(l_0 k) \left[\cos k(x_0 - r) - \cos k(x_0 + r)\right] dk. \tag{3.29}$$

If this expression is compared with (3.23), we find a fundamental difference in that D^α does not vanish at all points outside the light-cone. We cannot, therefore, satisfy the relation (3.23), but we can attain that D^α has essentially the same properties when a factor in an integral (as $\alpha \to 0$) by letting it behave as the half of (3.23) inside the light-cone. According to (3.24) we then have, in the vicinity of the direct cone mantle,

$$D^\alpha(\boldsymbol{x}) \sim \frac{\alpha}{4\pi} |R(\boldsymbol{x})|^{\alpha-1} w_{2\alpha}(\boldsymbol{x}) = \frac{\alpha}{4\pi} |x_0 - r|^{\alpha-1} |x_0 + r|^{\alpha-1} w_{2\alpha}(\boldsymbol{x}) =$$

$$= \frac{\alpha}{8\pi r} |x_0 - r|^{\alpha-1} f_\alpha(\boldsymbol{x}), \tag{3.30}$$

where $f_\alpha(\boldsymbol{x})$, of dimension $l_0^{-\alpha}$, is

$$f_\alpha(\boldsymbol{x}) = \frac{2r}{(x_0 + r)^{1-\alpha}} w_{2\alpha}(\boldsymbol{x}). \tag{3.31}$$

Our earlier condition on $w_{2\alpha}$ (section 4, Eq. (1.29)) is translated into the same condition on f_α: it is to approach unity continuously in α and in x_ν as $\alpha \to 0$, $x_0 \to r$. Going back to (3.29), we see that $w_{2\alpha}$ must here be chosen so as to make f_α a function of $|x_0 - r|$ only, which we may write

$$f_\alpha(\boldsymbol{x}) = l_0^{-\alpha} h_\alpha\left(\frac{x_0 - r}{l_0}\right). \tag{3.32}$$

Thus in the special coordinate system defined above[1]

$$D^{\alpha}(\boldsymbol{x}) = \frac{\alpha}{8\pi r} l_0^{-\alpha} \left[|x_0 - r|^{\alpha-1} h_\alpha \left(\frac{x_0 - r}{l_0} \right) - |x_0 + r|^{\alpha-1} h_\alpha \left(\frac{x_0 + r}{l_0} \right) \right], \tag{3.33}$$

so that, putting $\xi = (x_0 \pm r)/l_0$,

$$\frac{2}{\pi} \int_0^\infty g_\alpha(l_0 k) \cos(l_0 k \xi) \, d(l_0 k) = \alpha \, \xi^{\alpha-1} h_\alpha(\xi) \tag{3.34}$$

(for $\xi > 0$). By means of a Fourier transformation,

$$g_\alpha(l_0 k) = \alpha \int_0^\infty \xi^{\alpha-1} h_\alpha(\xi) \cos(l_0 k \xi) \, d\xi. \tag{3.35}$$

Eq. (3.33) gives the right behaviour, for if we write

$$D^{\alpha}(\boldsymbol{x}) = \frac{\alpha}{4\pi} |R(\boldsymbol{x})|^{\alpha-1} \bar{w}_\alpha(\boldsymbol{x}), \tag{3.36}$$

then $\bar{w}_\alpha$ tends to one on the direct light-cone and to minus one on the retrograde cone, for $\alpha \to 0$.

From the definition and our earlier conditions it follows that $h_\alpha(\xi) \to 1$ continuously as $\alpha \to 0$, $\xi \to 0$; further $h_\alpha(\xi) = h_\alpha(-\xi)$. To settle all convergence questions at infinity we may also suppose that $h_\alpha(\xi)$ tends sufficiently rapidly to zero for large ξ's and for $\alpha > 0$, in a manner that we need not specify here. Thus it will not be sufficient to choose e.g. $h_\alpha(\xi) = 1$, but we may take it to be proportional to $e^{-\alpha\xi}$; the exact behaviour is not important because in the end results it is only the form of the function for $\alpha \to 0$, $\xi \to 0$ that plays any role. However, the consequences of the theory will turn out not to be uniquely determined by the condition $h_0(0) = 1$; the derivatives at $\xi = 0$ will also be of importance. From the connection with Riesz's method it would be most natural to have the derivatives of $w_{2\alpha}$ vanish on the cone mantle, but as is seen from (3.31) and (3.32) this cannot be attained. To fix the problem we may therefore lay down that $h_\alpha(\xi)$, besides

$$h_0(0) = 1, \tag{3.37}$$

shall also conform to the condition

$$\frac{d^n h_0(0)}{d\xi^n} = 0 \quad (n = 1, 2, 3, \ldots). \tag{3.38}$$

We shall not use this condition, however, until later, section 17.

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[1] Cf. that $u = \frac{1}{r}[F(x_0 - r) - F(x_0 + r)]$ is the general form of a spherically symmetrical function which everywhere, also at the origin, satisfies $\Box\, u = 0$.

In section 20 we shall also be obliged to make some assumption of how h_α behaves as a function of α near $\alpha = 0$.

As to $g_\alpha(l_0 k)$, it follows immediately from (3.35) and (3.37) (cf. the lemma (1.32)) that

$$g_\alpha(l_0 k) \to 1 \text{ for } \alpha \to 0. \tag{3.39}$$

Moreover, if $h_\alpha(\xi) \sim 1$ for small values of ξ, $g_\alpha(l_0 k)$ will behave essentially as $(l_0 k)^{-\alpha}$ for large values of k. We shall come back to other properties of g_α in section 17.

14. Equivalence with the Riesz method. Let us now return to the Hamiltonian formalism for the electron. We have a theory holding for general α's, but it is essentially non-relativistic because of the special part played by the direction of $\boldsymbol{l}$. The non-relativistic character is also brought to light by the fact that D^α does not vanish outside the light-cone (when $\alpha > 0$), except on a three-dimensional plane through the origin orthogonal to $\boldsymbol{l}$; only on this plane is the compatibility relation (3.11) satisfied. However, if we let $h_\alpha(\xi)$ go down to zero for ξ greater than some value ξ_0, D^α can be made to vanish everywhere except in the vicinity of the cone mantle.

The canonical scheme given in section 12 is in so far incomplete that no Hamiltonian is defined for the field, only for the electrons and their interaction with the field. This gap will be filled in section 16.

We have arrived at D^α through a modification of the Riesz kernel, but we have yet to verify that the same results are obtained. To do this we make an analytic continuation to $\alpha = 0$ in Eq. (3.15), after first transforming the integral in the manner of (1.31). Writing the integration point as $\boldsymbol{z}'$ we have (from (3.36))

$$\begin{aligned} A_\mu(\boldsymbol{x}, s) &= -\frac{e}{\varepsilon_0} \int_{z_\mu(-\infty)}^{z_\mu(s)} D^\alpha(\boldsymbol{x} - \boldsymbol{z}')\, dz'_\mu = \\ &= \frac{e}{4\pi\varepsilon_0}\, \alpha \int_{R(x-z)}^{\infty} |R|^{\alpha-1}\, \bar{w}_\alpha(\boldsymbol{x} - \boldsymbol{z}') \frac{\partial z'_\mu}{\partial R}\, dR. \end{aligned} \tag{3.40}$$

If $\boldsymbol{x} - \boldsymbol{z}$ is space-like, so that $R(\boldsymbol{x} - \boldsymbol{z}) < 0$, this gives for $\alpha \to 0$

$$A_\mu(\boldsymbol{x}) = \frac{e}{2\pi\varepsilon_0} \left\{\frac{\partial z_\mu}{\partial R}\right\}_{R=0} \tag{3.41}$$

as in (1.34). As before, the derivatives are also seen to come out correct at points outside the world-line.

It remains to evaluate the analytic continuation of the field at the point $\boldsymbol{z}$ on the world-line. To this purpose we rewrite Eq. (3.40) by introducing $\tau = s - s'$ as integration variable:

$$A_\mu(\boldsymbol{z}, s) = -\frac{e}{\varepsilon_0}\int_0^\infty D^\alpha(\boldsymbol{z}-\boldsymbol{z}')\, u'_\mu\, d\tau =$$

$$= -\frac{e}{4\pi\varepsilon_0}\,\alpha\int_0^\infty R(\boldsymbol{z}-\boldsymbol{z}')^{\alpha-1}\,\bar{w}_\alpha(\boldsymbol{z}-\boldsymbol{z}')\, u'_\mu\, d\tau. \tag{3.42}$$

The derivatives of this expression can be obtained by differentiations under the sign of integration. For $\alpha \to 0$ it is only the neighbourhood of $\tau = 0$ that is of importance, and therefore it is sufficient to extend the integral over a small interval $(0, T)$, in which we may assume the integrand to be expansible in powers of τ. Further it is a consequence of the lemma (1.32), (1.33) that the analytic continuation of (3.42), as of all derivatives, is unaltered if the coefficients in the expansion are replaced by their values for $\alpha = 0$. We can use this to simplify $\bar{w}_\alpha$.

With

$$l_0\,\xi = z_0 - z_0' + r, \quad l_0\,\eta = z_0 - z_0' - r, \quad \text{i.e. } 2r = l_0(\xi - \eta) \tag{3.43}$$

(where now $r = |\vec{z} - \vec{z}'|$), we have from (3.33) and (3.36), and since η is here positive,

$$\bar{w}_\alpha(\boldsymbol{z}-\boldsymbol{z}') = l_0^{-2\alpha}\,\frac{\xi^{1-\alpha}\,h_\alpha(\eta) - \eta^{1-\alpha}\,h_\alpha(\xi)}{\xi - \eta}. \tag{3.44}$$

ξ and η are both of the order of magnitude of τ or of $R^{1/2}$ (= the Lorentz distance): if $u = |\vec{u}|$, so that $u_0^2 - u^2 = 1$, we have according to (3.43)

$$\begin{aligned} l_0\,\xi &= (u_0 + u)\,R^{1/2}(1 + a_1 R^{1/2} + \cdots), \\ l_0\,\eta &= (u_0 - u)\,R^{1/2}(1 - a_1 R^{1/2} + \cdots). \end{aligned} \tag{3.45}$$

Using $O(\alpha)$ to denote a quantity that vanishes of order α at least (note 1 p. 380), we then get

$$\xi^{1-\alpha} = l_0^\alpha R^{-\alpha/2}\,\xi + O(\alpha), \quad \eta^{1-\alpha} = l_0^\alpha R^{-\alpha/2}\,\eta + O(\alpha). \tag{3.46}$$

Finally $h_\alpha(\xi)$ and $h_\alpha(\eta)$ are expanded in Taylor series:

$$\begin{aligned} h_\alpha(\xi) &= h_\alpha(0) + \xi\,h'_\alpha(0) + \cdots \\ &= 1 + \xi\,h'_0(0) + \tfrac{1}{2}\xi^2\,h''_0(0) + \cdots + O(\alpha) \end{aligned} \tag{3.47}$$

etc. Then the numerator in (3.44) contains the factor $\xi - \eta$, which can be cancelled against the denominator. The result is that

$$\bar{w}_\alpha(\boldsymbol{z}-\boldsymbol{z}') = l_0^{-\alpha} R^{-\alpha/2}\left[1 - \tfrac{1}{2}\,\xi\,\eta\,h''_0(0) - \tfrac{1}{6}\,\xi\,\eta\,(\xi+\eta)\,h'''_0(0) - \cdots\right] + O(\alpha). \tag{3.48}$$

Apart from terms that vanish with α, the successive derivatives $\partial \bar{w}_\alpha(\boldsymbol{z}-\boldsymbol{z}')/\partial z_\nu$ etc. can obviously be obtained by differentiations of (3.48). Introduced into (3.42) the expression (3.48) gives, with $l_0^2\,\xi\,\eta = R$,

$$A_\mu(\boldsymbol{z}, s) = -\frac{e}{4\pi\varepsilon_0}\, l_0^{-\alpha}\,\alpha \int_0^T \left[R^{\frac{\alpha}{2}-1} - \frac{l_0^{-2}}{2}\, h_0''(0)\, R^{\frac{\alpha}{2}} - \cdots\right] u'_\mu\, d\tau + O(\alpha), \tag{3.49}$$

which can still be differentiated under the integral sign (we have $R = R(\boldsymbol{z}-\boldsymbol{z}')$).

The first term is the same as in the Riesz method and thus gives the same value for $\alpha \to 0$; the higher terms vanish in the limit. This is true also for the derivatives. Thus, since $\partial R/\partial z^\nu = -2\,(z_\nu - z'_\nu)$,

$$\frac{\partial A_\mu(\boldsymbol{z}, s)}{\partial z^\nu} = -\frac{e}{2\pi\varepsilon_0}\, l_0^{-\alpha}\,\alpha \int_0^T R^{\frac{\alpha}{2}-2}\,(z_\nu - z'_\nu)\, u'_\mu\, d\tau + O(\alpha), \tag{3.50}$$

which, according to

$$\begin{aligned} \boldsymbol{u}' &= \boldsymbol{u} - \dot{\boldsymbol{u}}\,\tau + \tfrac{1}{2}\,\ddot{\boldsymbol{u}}\,\tau^2 - \cdots, \\ \boldsymbol{z} - \boldsymbol{z}' &= \boldsymbol{u}\,\tau - \tfrac{1}{2}\,\dot{\boldsymbol{u}}\,\tau^2 + \cdots, \\ R &= \tau^2\,(1 + \tfrac{1}{12}\,\dot{\boldsymbol{u}}^2\,\tau - \cdots), \end{aligned} \tag{3.51}$$

gives (for $\alpha = 0$)

$$\frac{\partial A_\mu(\boldsymbol{z})}{\partial z^\nu} = -\frac{e}{4\pi\varepsilon_0}\,(\ddot{u}_\mu\, u_\nu + \tfrac{1}{3}\, u_\mu\, \ddot{u}_\nu + \dot{u}_\mu\, \dot{u}_\nu - \tfrac{1}{3}\, \dot{\boldsymbol{u}}^2\, u_\mu\, u_\nu), \tag{3.52}$$

as does the Riesz method.[1]

It is easy to see that this equivalence is general, so that all derivatives coincide, the terms in (3.49) with h_0'', . . . being unimportant for $\alpha \to 0$. This follows from the fact that a general term in the bracket expression in (3.49) is of the form

$$-\frac{l_0^{-2}}{(n+2)!}\, h_0^{(n+2)}(0)\, R^{\frac{\alpha}{2}}\,(\xi^n + \xi^{n-1}\,\eta + \cdots + \eta^n), \tag{3.53}$$

which because of the symmetry can be expressed as a polynomial in $l_0^2\,\xi\,\eta = R$ and $l_0\,(\xi + \eta) = 2\,(z_0 - z'_0)$:

$$\sum_{m=0}^{[n/2]} A_m\, R^{\frac{\alpha}{2}+m}\,(z_0 - z'_0)^{n-2m}. \tag{3.54}$$

By differentiations with respect to the z_μ's it is not possible to lower the degree so as to make the expression proportional to $\tau^{\alpha-1}$, $\tau^{\alpha-2}$, etc., without introducing a factor α as a coefficient, and therefore the assertion becomes clear.

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[1] See MA [1].

We have thus proved the complete equivalence between the present formalism and Riesz's method in the case of a classical point electron. In particular, the equation of motion (3.10) is the same as the earlier equation (1.18). We may remark that though it was not necessary to postulate the validity of (3.38), Eq. (3.48) shows that we get a closer agreement with the original Riesz kernel near the world-line (not outside it) if that relation is fulfilled.

15. Connection with the λ-limiting process. In the last section the connection between the present canonical formalism and the Riesz method was stressed. It is interesting that the canonical form of the α-method bears also a close resemblance to the λ-process. To see this we have only to return to expression (3.35) for $g_\alpha(\boldsymbol{l}, \boldsymbol{k})$, where the cosine factor is of the same form as in the λ-process; insertion into $D^\alpha(\boldsymbol{x})$, Eq. (3.25), then gives, on formal inversion of the order of integration,

$$D^\alpha(\boldsymbol{x}) = \alpha \int_0^\infty d\xi\, \xi^{\alpha-1} h_\alpha(\xi) \cdot \tfrac{1}{2}\left[D(\boldsymbol{x} + \xi \boldsymbol{l}) + D(\boldsymbol{x} - \xi \boldsymbol{l})\right]. \tag{3.55}$$

Here $\xi \boldsymbol{l}$ corresponds to the vector λ in the λ-process; and as $\alpha \to 0$ the values of $\xi l_0 = \lambda$ that are essential in the integral will also tend to zero. From (3.55) it is evident that *in all cases where the λ-process yields finite values, the present formulation of the α-method will give the same results.* In particular this is true of all field quantities on the world-line, which must consequently appear as half the difference of the retarded and advanced fields. From our comparison in section 14 with the Riesz method, it now follows that this method and the λ-limiting process are equivalent in applications to classical electrodynamics, as already shown by Ma[1].

The usual expression for the D function in the λ-process appears if we choose $h_\alpha(\xi) = \xi^{-\alpha}\delta(\xi - a)$, which is of course not of the form required to get a connection with the Riesz method. Instead the form (3.55) corresponds to a certain averaging process (to be followed by an analytic continuation), by which the power of the method is increased[2] (see section 17).

Quantum theory

16. Formulation of the equations. Now that a Hamiltonian formalism has been developed for general α's in the classical theory, it can be immediately transformed to the quantum theory by using the connection between the classical Poisson bracket $[A, B]$ and the commutator $[A, B] = AB - BA$ in quantum mechanics[3]:

$$[A, B]_{\text{quant}} \leftrightarrow i\hbar\,[A, B]_{\text{class}}. \tag{3.56}$$

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[1] Ma [1].

[2] The possibility of such a generalization of the λ-process, by combining it with analytic continuation, has already been expressed by Ma (unpublished).

[3] See Dirac [5] § 21.

The scheme of equations given in section 11 is then unchanged but for the D function being replaced by D^α. Thus Eq. (3.2), $\Box A_\mu = 0$, is the same equation as (3.22) in the classical case, while the classical relation (3.14) is seen from (3.56) to correspond to

$$[A_\mu(\boldsymbol{x}), A_\nu(\boldsymbol{y})] = -g_{\mu\nu} \frac{i\hbar c}{\varepsilon_0} D^\alpha(\boldsymbol{x}-\boldsymbol{y}), \tag{3.57}$$

and Eq. (3.18) to

$$\left[\frac{\partial A_\mu(\boldsymbol{x})}{\partial x_\mu} - \sum \frac{e}{\varepsilon_0} D^\alpha(\boldsymbol{x}-\boldsymbol{z})\right] |\boldsymbol{z}\rangle = 0. \tag{3.58}$$

The consistency of the scheme follows from the corresponding considerations in the classical case.

It is easy to express the formalism in momentum space. We resolve $A_\mu(\boldsymbol{x})$ into Fourier components:

$$A_\mu(\boldsymbol{x}) = \frac{(2\pi)^{-3/2}}{\sqrt{2}} \int \left[A_\mu(\boldsymbol{k})\, e^{i(\boldsymbol{k},\boldsymbol{x})} + A_\mu^*(\boldsymbol{k})\, e^{-i(\boldsymbol{k},\boldsymbol{x})}\right] \frac{(d\,k)^3}{k} \tag{3.59}$$

with $k_0 = k$, i. e. $(\boldsymbol{k}, \boldsymbol{k}) = 0$, which corresponds to Eq. (3.2). The commutation relations (3.57), combined with the expression (3.27) for D^α, then show that

$$[A_\mu(\boldsymbol{k}), A_\nu^*(\boldsymbol{k}')] = g_{\mu\nu} \frac{\hbar c}{\varepsilon_0} k\, g_\alpha(\boldsymbol{l}, \boldsymbol{k})\, \delta(\vec{k}-\vec{k}'), \tag{3.60}$$

with other commutators zero. Finally, from (3.58),

$$\begin{aligned}
&\left[k^\mu A_\mu(\boldsymbol{k}) + \frac{(2\pi)^{-3/2}}{\sqrt{2}} g_\alpha(\boldsymbol{l}, \boldsymbol{k}) \sum \frac{e}{\varepsilon_0} e^{-i(\boldsymbol{k},\boldsymbol{z})}\right] |\boldsymbol{z}\rangle = 0, \\
&\left[k^\mu A_\mu^*(\boldsymbol{k}) + \frac{(2\pi)^{-3/2}}{\sqrt{2}} g_\alpha(\boldsymbol{l}, \boldsymbol{k}) \sum \frac{e}{\varepsilon_0} e^{i(\boldsymbol{k},\boldsymbol{z})}\right] |\boldsymbol{z}\rangle = 0.
\end{aligned} \tag{3.61}$$

For the sake of completeness we give the expression for the Hamiltonian of the electromagnetic field (with the zero-point energy subtracted) for general α's, formed in the same way as done by Pauli[1] in the case of the λ-limiting process:

$$H^F = \varepsilon_0 \int \left[\vec{A}^*(\boldsymbol{k})\, \vec{A}(\boldsymbol{k}) - A_0(\boldsymbol{k})\, A_0^*(\boldsymbol{k})\right] g_\alpha(\boldsymbol{l}, \boldsymbol{k})^{-1} (d\,k)^3. \tag{3.62}$$

From (3.62) and the relations

$$\begin{aligned}
[A_\mu(\boldsymbol{x}), A_\nu^*(\boldsymbol{k})] &= g_{\mu\nu} \frac{(2\pi)^{-3/2}}{\sqrt{2}} \frac{\hbar c}{\varepsilon_0} g_\alpha\, e^{i(\boldsymbol{k},\boldsymbol{x})}, \\
[A_\mu(\boldsymbol{x}), A_\nu(\boldsymbol{k})] &= -g_{\mu\nu} \frac{(2\pi)^{-3/2}}{\sqrt{2}} \frac{\hbar c}{\varepsilon_0} g_\alpha\, e^{-i(\boldsymbol{k},\boldsymbol{x})},
\end{aligned} \tag{3.63}$$

---

[1] PAULI [3] § 6 (a).

which are a consequence of Eqs. (3.59) and (3.60), we conclude

$$\frac{\hbar}{i}\,\frac{\partial A_\mu(\boldsymbol{x})}{\partial x_0} = \frac{1}{c}\,[H^F, A_\mu(\boldsymbol{x})], \tag{3.64}$$

so that (3.62) is the correct definition.

The formalism being non-relativistic for general α's, it is natural to use a one-time formulation in the presence of many electrons. If the operator in (3.1), with $z_\mu = z_\mu^{(n)}$, $\beta = \beta^{(n)}$, etc., is written $-i\hbar\,\partial/\partial z_0^{(n)} + H^{(n)}/c$, we obtain after putting all $z_0^{(n)} = z_0$ in $|\boldsymbol{z}\rangle = |\boldsymbol{z}^{(1)}, \ldots\rangle$

$$\left[\frac{\hbar}{i}\,\frac{\partial}{\partial z_0} + \frac{1}{c}\sum_n H^{(n)}\right]|\vec{z}^{(1)}, \ldots; z_0\rangle = 0. \tag{3.65}$$

The coordinate system employed should have its time axis in the direction of $\boldsymbol{l}$, so that the plane $z_0 = \text{const.}$ is orthogonal to $\boldsymbol{l}$ (cf. the first paragraph of section 14 and section 25).

The simplification of the equation by using the supplementary condition (3.58) or (3.61) to eliminate the longitudinal waves follows the usual scheme[1] and need not be repeated here. The resulting Coulomb term (electrostatic self-energy) is proportional to the analytic continuation of the integral $\int\limits_0^\infty g_\alpha(l_0 k)\,dk$, and we are therefore interested in knowing the properties of g_α as a factor in an integral.

17. **Properties of g_α.** Starting from the definition (3.35) we have found in (3.39) that $g_\alpha \to 1$ for $\alpha \to 0$. If the same expression is used again we find, on formally reversing the order of integration,

$$\int\limits_0^\infty k^\beta g_\alpha(l_0 k)\,dk = \alpha \int\limits_0^\infty d\xi\,\xi^{\alpha-1} h_\alpha(\xi) \int\limits_0^\infty k^\beta \cos(l_0 k \xi)\,dk =$$

$$= -\,l_0^{-1-\beta}\,\Gamma(1+\beta)\,\sin\frac{\beta\pi}{2}\,\alpha \int\limits_0^\infty \xi^{\alpha-\beta-2} h_\alpha(\xi)\,d\xi. \tag{3.66}$$

It is true that the integral over k in the middle term is properly convergent only for $-1 < \beta < 0$, but with a suitable h_α it is evident that we can choose α large enough for the first member in (3.66) to be convergent for an arbitrary fixed $\beta > -1$, as also the last member; since these two expressions are moreover analytic in β, they must be equal for every $\beta > -1$.

We now want to make an analytic continuation to $\alpha = 0$. In the last integral only the neighbourhood of $\xi = 0$ is then important (the integral converges at $\xi = \infty$, also when $\alpha = 0$, provided $h_\alpha(\xi) = O(\xi^{-\alpha})$ as $\xi \to \infty$), and we can therefore write

---

[1] FERMI [1]; see e. g. PAULI [3] § 6 (a), WENTZEL [2] §§ 17, 19, or DIRAC [5] § 80.

$$\operatorname*{cont}_{\alpha=0}\int_0^\infty k^\beta g_\alpha\,d\,k = -\,l_0^{-1-\beta}\,\Gamma(1+\beta)\,\sin\frac{\beta\,\pi}{2}\operatorname*{cont}_{\alpha=0}\,\alpha\int_0^\delta \xi^{\alpha-\beta-2}\,h_\alpha(\xi)\,d\,\xi =$$

$$= -\,l_0^{-1-\beta}\,\Gamma(1+\beta)\,\sin\frac{\beta\,\pi}{2}\operatorname*{cont}_{\alpha=0}\sum_n \frac{h_\alpha^{(n)}(0)}{n!}\,\alpha\int_0^\delta \xi^{\alpha+n-\beta-2}\,d\,\xi, \quad (3.67)$$

or as is easily seen from the lemma (1.32), (1.33)

$$\operatorname*{cont}_{\alpha=0}\int_0^\infty k^\beta g_\alpha(l_0\,k)\,d\,k =$$

$$= \begin{cases} l_0^{-2n-2}\dfrac{(-1)^{n+1}}{2\,n+2}\,h_0^{(2n+2)}(0) & \text{for } \beta = 2\,n+1,\ n = 0, 1, 2, \ldots \\ 0 & \text{for other } \beta\text{'s} > -1. \end{cases} \quad (3.68)$$

To make the expression well defined for odd integral values of β we must fix the values of the derivatives of $h_0(\xi)$ at $\xi = 0$. The simplest choice is to make them equal to zero, i. e. to assume (3.38). Then

$$\operatorname*{cont}_{\alpha=0}\int_0^\infty k^\beta g_\alpha(l_0\,k)\,d\,k = 0 \qquad \text{for } \beta > -1. \quad (3.69)$$

This is in fact the relation required by Pauli, see [II] § V.

The exact form of g_α is immaterial. Pauli's proposal is

$$g_\alpha(l_0\,k) = \begin{cases} (l_0\,k)^{-\alpha} & \text{for } l_0\,k \geq \varepsilon, \\ \varepsilon^{-\alpha} & \text{for } l_0\,k < \varepsilon. \end{cases} \quad (3.70)$$

That g_α behaves as $(l_0\,k)^{-\alpha}$ for large values of k, if $h_\alpha(\xi) \sim 1$ for small ξ's, has already been remarked. We can obtain other possible functions g_α by starting from a function $h_\alpha(\xi)$ that conforms to (3.37) and (3.38), besides tending to zero more rapidly than $\xi^{-\alpha}$ for $\xi \to \infty$. Thus with $h_\alpha(\xi) = e^{-a\,\xi}/\Gamma(1+a)$,

$$g_\alpha(l_0\,k) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-a\,\xi}\,\xi^{\alpha-1}\cos(l_0\,k\,\xi)\,d\,\xi =$$

$$= \cos\left(\alpha\arctan\frac{l_0\,k}{a}\right)(l_0^2\,k^2 + a^2)^{-\frac{\alpha}{2}}. \quad (3.71)$$

Since it is desirable to let g_α be a positive function of k[1], it would be more convenient to use only the last factor in (3.71) (which determines the behaviour for large k's):

$$g_\alpha(l_0\,k) = (l_0^2\,k^2 + a^2)^{-\alpha/2}. \quad (3.72)$$

---

[1] Otherwise certain complications will appear in characterizing $A_\mu(\boldsymbol{k})$ and $A_\mu^*(\boldsymbol{k})$ as absorption and emission operators; cf. in the λ-process the third edition of DIRAC [5] § 79.

However, this question is not important, because it is only the fact of relation (3.69) being fulfilled that has to be exploited in the applications.

It is natural, from the discussion in section 15, that in those cases where the integral (3.68) is left ambiguous[1] unless the derivatives $h_0^{(n)}(0)$ are fixed, the λ-limiting process does not attain convergence at all. The fact that we can make (3.69) hold for all β's (> -1) shows that *the α-continuation method is more powerful in eliminating divergences than the λ-limiting process.*

18. A further connection with Riesz's integration method. The formalism developed in this chapter may be said to consist in a modification of the pure vacuum field ($\Box A_\mu = 0$) by means of a parameter α. Now we had already made such a modification in section 8, where the vacuum field was treated by applying Riesz's method to the Cauchy problem involving a (variable) surface S. As a matter of fact the expression $\mathcal{A}_\mu^{(\alpha)}$ obtained there and our present A_μ coincide if we let

$$g_\alpha(l_0 k) = l_0^{-2\alpha} F_\alpha(k, l_0) F_\alpha^*(k, l_0), \tag{3.73}$$

which is defined by (2.37). It is interesting that this definition agrees with relation (3.69), as shown by Eq. (2.40).

19. Applications to self-energy calculations. We can now resume the discussion left off at the end of section 16. It follows from (3.69) that the electrostatic self-energy vanishes, being proportional to cont $\int_0^\infty g_\alpha k^n \, dk$ with $n = 0$. If the equation for $|z\rangle$ is then treated by perturbation methods, the lowest approximation of the electrodynamic self-energy is obtained in the same form, with $n = 1$, and so vanishes too.

Thus, as in the case of Gustafson's method, the value zero is found for the self-energy of an electron at rest in the second-order approximation (on the one-electron theory) when Pauli's form of the α-method is used. Quite generally, Gustafson's and Pauli's formulations remove the same divergences, though the results need not always coincide, as was shown in [II].

The explicit calculation of the self-energy is to be found in [III] (where φ_α stands for our present g_α, $M = l_0^{-1}$) and is there also continued to the next approximation, of the fourth order in the interaction constant e. This approximation turns out to be infinite, owing to the appearance of logarithmic divergences. It may be mentioned as a comparison that, using the assumption of negative-energy photons, Eliezer could make all higher approximations vanish.[2]

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[1] Cf. the non-uniqueness in Pomeranchuk's method, POMERANCHUK [1].

[2] ELIEZER [1]; a slight correction of a step in the calculation is given in ELIEZER [2] Chap. V.

CHAPTER IV

Logarithmic divergences

20. A modification of the scheme. Whereas the α-method could eliminate the divergences in the lowest approximation on the one-electron theory, the logarithmic divergences of the hole theory have remained unaltered. It was remarked in [II] that these can be removed too if Pauli's method is formulated in terms of several parameters[1] $\alpha_1, \alpha_2, \ldots$, with $g_{\alpha_1 \ldots}(l_0 k) \sim \sum_n b_n(\alpha_n)(l_0 k)^{-\alpha_n}$, but the value of such generalizations is impaired by the fact that it is always possible to choose b_n and α_n in different ways so as to get quite different numerical results. However, we want to dwell upon a particularly simple formulation proposed by Riesz[2], which can be expressed with the help of one parameter α.

We shall here arrive at this formulation in the following way. Without expressly saying so, we have up to now regarded α as an essentially real parameter; at least it is evident that if $A_\mu(\boldsymbol{x})$ is to represent a real field for general α's, the factor g_α in the commutator (3.60) must be real. If complex values of α are allowed, we therefore write

$$[A_\mu(\boldsymbol{k}), A_\nu^*(\boldsymbol{k}')] = g_{\mu\nu} \frac{\hbar c}{\varepsilon_0} k\, G_\alpha(l_0 k)\, \delta(\vec{k} - \vec{k}') \tag{4.1}$$

with

$$G_\alpha(l_0 k) = \tfrac{1}{2}[g_\alpha(l_0 k) + g_\alpha^*(l_0 k)], \tag{4.2}$$

i.e. G_α is the real part of g_α. Then G_α is not an analytic function of α; in reality it depends on two parameters α and α' with $\alpha' = \alpha^*$. We intend to show that logarithmic singularities of the type $\int^\infty dk/k$ are eliminated if the continuation to $\alpha = 0$ is performed along a path which, starting from values of α with a positive real part, ends by approaching the origin along the imaginary axis. That is indeed the prescription given by Riesz. We shall indicate this type of path by the notation $\alpha \to i \cdot 0$.

We first note that the properties of g_α in integral expressions established in section 17 are unaltered by the new prescription. Thus it follows immediately from (3.69) and (4.2) that

$$\operatorname*{cont}_{\alpha \to i0} \int_0^\infty k^\beta G_\alpha(l_0 k)\, dk = 0 \quad \text{for} \quad \beta > -1. \tag{4.3}$$

Our earlier results are therefore unaffected by the change, in so far as they do not involve logarithmic divergences.

To see the effect of using G_α instead of g_α in logarithmic expressions, we turn to the behaviour of an integral $\int^\infty g_\alpha\, dk/k$ as $\alpha \to 0$. With the simple choice $g_\alpha = (l_0 k)^{-\alpha}$ for large k's we find

---

[1] This formulation follows a proposal by Pomeranchuk for the λ-process, POMERANCHUK [1].

[2] I am indebted to Professor Riesz for this suggestion.

$$\int_K^\infty g_\alpha \frac{dk}{k} = \int_K^\infty (l_0 k)^{-\alpha} \frac{dk}{k} = \frac{(l_0 K)^{-\alpha}}{\alpha} = \frac{1}{\alpha} - \log (l_0 K) + O(\alpha) \tag{4.4}$$

and thus

$$\operatorname*{cont}_{\alpha \to i0} \int_K^\infty G_\alpha (l_0 k) \frac{dk}{k} = - \log (l_0 K), \tag{4.5}$$

since $1/\alpha + 1/\alpha^* \to 0$ when α approaches the origin along the particular path $\alpha \to i0$.

We shall not reproduce the calculation for the case that a more general expression for g_α is used, of the form (3.35), only remarking that there may then come in additive terms from the behaviour of h_α as a function of α; to obtain (4.5) one must require that[1]

$$\left[\frac{\partial h_\alpha (0)}{\partial \alpha}\right]_{\alpha=0} = C, \tag{4.6}$$

C being Euler's constant. Another value of the derivative would only correspond to a renormalization of l_0 in (4.5); the value must be fixed, however.

21. Modification of the Riesz-Gustafson method. We can combine the prescription of the last section also with the formulation in Chapter II. For complex values of α we then have to replace

$$V^\alpha \to \tfrac{1}{2} (V^\alpha + V^{\alpha *}) \tag{4.7}$$

and correspondingly for $A_\mu^{(\alpha)}$; α is then to go to zero along the path $\alpha \to i0$.

We may note in passing that for complex values of α the denominator $[(\boldsymbol{K}, \boldsymbol{K}) + \alpha^2/l_0^2]^{1+\alpha/2}$ mentioned in section 8 does not vanish for any real value of $(\boldsymbol{K}, \boldsymbol{K})$; therefore the kernel $\frac{1}{2} (V_{\alpha/l_0}^{\alpha+2} + \text{conj.})$ will take away the unessential singularities that may appear when $V^{\alpha+2}$ is used.

22. Examples: Self-energy in the hole theory. As an application of the formalism in section 20 we turn to the self-energy of an electron at rest calculated on the basis of the hole theory. On specializing the units,

$$\varepsilon_0 = 1/4\pi, \hbar = c = 1, \tag{4.8}$$

one obtains an expression composed of the two terms[2]

---

[1] Cf. that, apart from the factor $2^{-\alpha}$, $1/H(\alpha+2)$ in the Riesz method is of the form $\frac{\alpha}{4\pi}(1 + C\alpha + \cdots)$.

[2] Cf. the corresponding expression obtained with the λ-process, PAULI [3] § 6 (b), Eqs. (191), (192).

$$E_{\text{st}} = \operatorname*{cont}_{\alpha \to i0} \frac{e^2 m}{\pi} \int_0^\infty G_\alpha (l_0 k) \frac{dk}{(k^2 + m^2)^{1/2}}, \tag{4.9}$$

$$E_{\text{dyn}} = \operatorname*{cont}_{\alpha \to i0} \frac{e^2}{\pi m} \int_0^\infty G_\alpha (l_0 k) \left[k - \frac{k^2}{(k^2 + m^2)^{1/2}} \right] dk. \tag{4.10}$$

Splitting up the first integral and using (4.5) we get

$$\operatorname*{cont}_{\alpha \to i0} \int_0^\infty G_\alpha \frac{dk}{(k^2 + m^2)^{1/2}} = \int_0^K \frac{dk}{(k^2 + m^2)^{1/2}} + \operatorname*{cont}_{\alpha \to i0} \int_K^\infty G_\alpha \frac{dk}{k} -$$

$$- \int_K^\infty \left[\frac{1}{k} - \frac{1}{(k^2 + m^2)^{1/2}} \right] dk = \log \frac{K + (K^2 + m^2)^{1/2}}{m} - \log l_0 K - \tag{4.11}$$

$$- \int_K^\infty \left[\frac{1}{k} - \frac{1}{(k^2 + m^2)^{1/2}} \right] dk,$$

where the analytic continuation has been performed under the integration sign in convergent integrals. The value of (4.11), being independent of K, can be obtained e. g. by combining the first two terms and letting $K \to \infty$:

$$\operatorname*{cont}_{\alpha \to i0} \int_0^\infty G_\alpha (l_0 k) \frac{dk}{(k^2 + m^2)^{1/2}} = - \log \frac{l_0 m}{2}. \tag{4.12}$$

In (4.10) the contribution to the integral from the first term vanishes on account of (4.3); the remaining part is written

$$E_{\text{dyn}} = \operatorname*{cont}_{\alpha \to i0} \frac{e^2 m}{\pi} \left[\int_0^\infty G_\alpha \frac{dk}{(k^2 + m^2)^{1/2}} - \frac{1}{m^2} \int_0^\infty G_\alpha (k^2 + m^2)^{1/2} \, dk \right]. \tag{4.13}$$

Through a similar decomposition to that in (4.11) it is found that

$$\operatorname*{cont}_{\alpha \to i0} \int_0^\infty G_\alpha (l_0 k) (k^2 + m^2)^{1/2} \, dk = \frac{m^2}{4} - \frac{m^2}{2} \log \frac{l_0 m}{2}. \tag{4.14}$$

Thus we get, on going back to electrostatic units ($\varepsilon_0 = 1/4\pi$),

$$E_{\text{st}} = - \frac{e^2}{\hbar c} \frac{m c^2}{\pi} \log \frac{l_0 m c}{2 \hbar}, \tag{4.15}$$

$$E_{\text{dyn}} = -\frac{e^2}{\hbar c}\,\frac{m\,c^2}{\pi}\left[\frac{1}{2}\log\frac{l_0\,m\,c}{2\,\hbar} + \frac{1}{4}\right]. \tag{4.16}$$

This corresponds to a mass-change $\varDelta\,m$ given by

$$\frac{\varDelta\,m}{m} = \frac{3}{2\,\pi}\,\frac{e^2}{\hbar\,c}\left[\log\frac{2\,\hbar}{l_0\,m\,c} - \frac{1}{6}\right]. \tag{4.17}$$

23. **Examples: Higher approximations.** Besides in the hole theory we have found logarithmic divergences in the higher approximations also in the one-electron theory, and one may ask oneself if these too can be eliminated by using the special formalism of section 20. That is not the case, however, owing to the more complicated form of the integral expressions in the higher approximations.

Indeed Riesz's formulation corresponds to writing

$$G_\alpha(l_0\,k) = \sum_n b_n\cdot(l_0\,k)^{-\alpha_n}\ \text{(for large } k\text{'s)} \tag{4.18}$$

with two parameters α_1 and $\alpha_2 = \alpha_1^*$ and with $b_1 = b_2 = 1/2$. That the logarithmic singularity in $\int^\infty d\,k/k$ is avoided by means of analytic continuation of $\int^\infty G_\alpha\,d\,k/k$ is a consequence of

$$\sum_n \frac{b_n}{\alpha_n} \to 0 \ \text{ for } \alpha_n \to 0. \tag{4.19}$$

If we use a general expression of the form (4.17), with $\sum b_n \to 1$ as α_1, $\alpha_2, \ldots \to 0$, it is found by repeating the calculation of [III] that convergence in the fourth-order approximation would require not only that the sum in (4.19) should tend to zero (or generally to a finite value), but also that the same thing must be true of

$$\sum_n \frac{b_n}{\alpha_n^2},\quad \sum_n\sum_m \frac{b_n\,b_m}{\alpha_n + \alpha_m},\quad \sum_n\sum_m \frac{b_n\,b_m}{\alpha_n\,\alpha_m\,(\alpha_n + \alpha_m)}. \tag{4.20}$$

To satisfy these conditions we must use more than two parameters, and then the naturalness and simplicity characterizing Riesz's formulation would be lost.

Thus the method is not capable of coping with the difficulties presented by the divergence of the higher approximations.

24. **Examples: The level shift — non-relativistically.** If the electron is treated non-relativistically, the problem of an electron moving in an external potential field is characterized by the Hamiltonian

$$H = H^F + \frac{1}{2\,m}(\vec{p} + e\,\vec{A})^2 + V = H^0 + W \tag{4.21}$$

with

$$H^0 = H^F + \frac{1}{2m}\vec{p}^{\,2} + V, \tag{4.22}$$

$$W = \frac{e}{m}\vec{p}\,\vec{A}\left(+\frac{e^2}{2m}\vec{A}^2\right). \tag{4.23}$$

The units have been chosen as in (4.8); $\vec{p} = p_r$ $(r = 1, 2, 3)$ stands for the operator $-i\,\partial/\partial x_r$ and $\vec{A} = A_r(\vec{x})$ is the (transverse) vector potential of the radiation field (with Hamiltonian H^F); $V = V(\vec{x})$ is the potential energy.

If we let $|n\rangle$ denote an eigenvector of $H^0 - H^F$ belonging to the eigenvalue E_n, the second-order correction to E_n coming from the perturbing term W is easily found to be, in the present theory,

$$\Delta E_n = -\frac{e^2}{4\pi^2 m^2}\sum_l \int \frac{(dk)}{k} G_\alpha d_{rs} \frac{\langle n|p_r e^{i\vec{k}\vec{x}}|l\rangle\langle l|p_s e^{-i\vec{k}\vec{x}}|n\rangle}{k + E_l - E_n} \tag{4.24}$$

(analytic continuation to $\alpha = 0$); $d_{rs} = \delta_{rs} - n_r n_s$ if $n_r = k_r/k$.

For a free electron (4.24) diverges logarithmically[1] for $G_\alpha = 1$. Using analytic continuation according to Riesz we obtain

$$\Delta E^0(\vec{p}) = \frac{8e^2}{3\pi}\log(2\,l_0\,m)\frac{\vec{p}^{\,2}}{2m} = -\frac{\Delta m}{m}\frac{p^2}{2m} \tag{4.25}$$

for a free electron of momentum $\vec{p}$. This would correspond to an experimental electron mass

$$m_e = m + \Delta m = m\left[1 - \frac{8e^2}{3\pi}\log(2\,l_0\,m)\right]. \tag{4.26}$$

If the electron moves in a Coulomb field, $V = -Ze^2/r$, expression (4.24) can be written in the form

$$\Delta E_n = \Delta E_n^0 + E_n', \tag{4.27}$$

where the first term is the same expression as (4.25) but with $p^2/2m$ replaced by $\langle n|\vec{p}^{\,2}/2m|n\rangle = -E_n$. As is easily seen, the effect of this term is to substitute the experimental mass m_e for the mechanical mass m in the expression for the undisturbed energy levels,

$$E_n = -\frac{m e^4}{2}\frac{Z^2}{n^2}. \tag{4.28}$$

The observable effect of the influence of the radiation would then be given by the second term, $\Delta E_n'$, which converges also without using analytic continuation.

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[1] WALLER [1].

However, in the integration over k in (4.24) one gets outside the range of validity of the non-relativistic approximation for large k's, where the relativistic expression is seen to diverge linearly in k rather than logarithmically (on the one-electron theory). Taking this roughly into account by neglecting the retardation in (4.24) (the exponential factors then cancel) one finds

$$\Delta E_n = -\frac{2e^2}{3\pi m^2} \sum_l \int_0^\infty dk \cdot k\, G_\alpha \frac{\langle n|p_r|l\rangle\langle l|p_r|n\rangle}{k + E_l - E_n}, \tag{4.29}$$

where the integration over the directions of $\vec{k}$ has already been performed, using $\oint d_{rs}\, d\omega = \delta_{rs}\, 8\pi/3$.

Bethe[1] has used (4.29) (apart, of course, from the factor G_α) to compute the electromagnetic level shift established experimentally by Lamb and Retherford.[2] After discarding the contribution from the free-electron self-energy in (4.29) he has then to make a cut-off in the k-integral in order to obtain a finite result.

When analytic continuation is used we have in (4.29) an integral of the form

$$\int_0^\infty \frac{k\, G_\alpha}{k+\mu}\, dk = \int_0^\infty G_\alpha \cdot dk - \mu \int_0^\infty \frac{G_\alpha}{k+\mu}\, dk. \tag{4.30}$$

The first term on the right, corresponding to the self-energy of a free electron, now vanishes on continuation to $\alpha = 0$. The second integral can be reduced to the form (4.5):

$$\begin{aligned} \operatorname*{cont}_{\alpha \to i0} \int_0^\infty \frac{G_\alpha\, dk}{k+\mu} &= \int_0^K \frac{dk}{k+\mu} + \operatorname*{cont}_{\alpha \to i0} \int_K^\infty G_\alpha \frac{dk}{k} - \int_K^\infty \left(\frac{1}{k} - \frac{1}{k+\mu}\right) dk = \\ &= \log\left|\frac{K+\mu}{\mu}\right| - \log l_0 K - \mu \int_K^\infty \frac{dk}{k(k+\mu)}, \end{aligned} \tag{4.31}$$

or

$$\operatorname*{cont}_{\alpha \to i0} \int_0^\infty \frac{G_\alpha(l_0 k)}{k+\mu}\, dk = -\log|l_0 \mu|. \tag{4.32}$$

Insertion into ΔE_n yields

$$\Delta E_n = \frac{2e^2}{3\pi m^2} \sum_l \langle n|p_r|l\rangle\langle l|p_r|n\rangle\, (E_l - E_n) \log \frac{l_0^{-1}}{|E_l - E_n|}, \tag{4.33}$$

which is of the same form as Bethe's expression; the agreement is complete if l_0 is chosen to be the Compton wave-length,

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[1] BETHE [1].

[2] LAMB-RETHERFORD [1].

$$l_0 = \frac{\hbar}{mc} \tag{4.34}$$

(in ordinary units).

The result (4.33) shows that the method we have used here is in a sense equivalent to a cut-off; this is also seen from e. g. (4.15) and (4.16). Indeed, the main interest of the considerations in the present section consists in their affording an illustration of how the method works and of how it can be used to define such cut-offs in a consistent way; the exact form (4.33) of the result is not very important in view of the non-relativistic approach and the other approximations made above. In particular the justification for using (4.29) is not clear when the upper limit of the integral is infinite.

The way to set about the relativistic calculation will be touched upon in section 25.

CHAPTER V

Concluding remarks

25. Moving electron. Relativistic level shift. In this last chapter we are going to collect some remarks on a few miscellaneous topics, which must here be treated rather cursorily. The first concerns the self-energy of an electron in motion evaluated by means of the formalism in Chapters III and IV — up to now our applications have only been concerned with an electron at rest. No new problems arise in the one-electron theory, where the second-order approximation of the self-energy is still found to come out as zero. In the hole theory, however, matters are not so simple, because the result of treating the equations by standard methods, combined with the prescription in section 20, has not the correct transformation properties; this is a well-known feature also of the ordinary (divergent) expression. Nor is it possible, within the frame of the formalism, to go over for the calculation to a coordinate system in which the electron is at rest, and then transform back afterwards.

The reason for this lies in the non-relativistic character of the canonical formalism in Chapter III. It was pointed out near the end of section 16 that we must work in a three-dimensional plane orthogonal to the vector $\boldsymbol{l}$, i. e. use a Lorentz system with the time axis along the direction of $\boldsymbol{l}$; in any other system the three-dimensional space would have different properties in different directions. Since the anisotropy disappears in the limit $\alpha = 0$, it does not really matter in ordinary expressions, but in logarithmic terms it does. Thus the expression (4.15), (4.16) for the self-energy of an electron at rest on the hole theory would be totally changed if another coordinate system had been employed in the evaluation (the time and space parts of $\boldsymbol{l}$ would come in separately). It is therefore necessary to keep to one fixed Lorentz frame during the calculations; the question is only how to do it in a consistent way that preserves the relativistic invariance of the results.

A possibility that is near at hand is a combination of the present formalism with Schwinger's invariant formulation of quantum electrodynamics.[1] Such a combination amounts to replacing the various D functions in Schwinger's treatment of the self-energy, SCHWINGER [1] II § 3, by the corresponding D^{α} functions; the procedure is otherwise the same (but all calculations must be performed in the special coordinate system defined above). Evidently the outcome will be that the finite expression (4.17) for $\Delta m/m$ will take the place of the infinite quantities in Schwinger's equations (3.97) and (3.98). The renormalization of the mass can be performed in the same way as done by Schwinger, but with the advantage that Δm is now finite. Since only the combination $m + \Delta m = m_e$ enters into the resulting equations, there will be no observational difference, however.

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[1] SCHWINGER [1].

In particular, if we want to make a relativistic calculation of the electromagnetic shift of energy levels in hydrogen (section 24) with the help of the α-formalism, it is necessary to combine it with some such prescription as Schwinger's, the result then being the same as is obtained on his theory.[1]

26. Applications to meson theory. The formulation of the α-method in Chapters I and II was founded on Riesz's solution of the equation $-\Box u = f$. The corresponding solution for the equation $(-\Box + \varkappa^2)\,u = f$ can be used to treat the meson case in a similar way; this has been done classically by Fremberg[2] and quantum-theoretically by Gustafson[3]; some formulas relating to this problem are also given in [II]. The effect with respect to the elimination of divergences is the same as in electrodynamics.

If we adopt instead the canonical formalism of Chapter III, we have to introduce a D^{α} function for the meson field corresponding to the function D^{α} in (3.27). In the meson case $(\boldsymbol{k}, \boldsymbol{k}) = -\varkappa^2$, so that $k_0 = (k^2 + \varkappa^2)^{1/2}$ (the meson mass being $\hbar\varkappa/c$); therefore, if g_{α} is again taken to be a function of the scalar product $(\boldsymbol{l}, \boldsymbol{k})$, we have

$$g_{\alpha}(l_0 k_0) \sim (l_0 k_0)^{-\alpha} = l_0^{-\alpha}(k^2 + \varkappa^2)^{-\alpha/2} \quad \text{(for large } k\text{'s)} \tag{5.1}$$

in the special coordinate system employed in Chapter III (for which $\vec{l} = 0$). This form is in analogy with Pauli's formulation of the λ-process for the meson case.[4]

As in section 15, the α-method can still be regarded as a generalization of the λ-process and so will give the same results in cases where the latter process attains convergence. This means, for instance, that the magnetic moment of the neutron will appear with the wrong sign.[5] In order to avoid wrong numerical values it is therefore again necessary to resort to some renormalization procedure in the sense of Schwinger.[6]

27. Another use of analytic continuation. We shall now briefly return to the problem treated in Chapter I, that of deriving the equation of motion (1.18). Our present derivation will start from the ordinary retarded field outside the world-line, the value on the world-line being defined as the finite part of the field as one approaches the line orthogonally. Though more artificial, this mode of procedure shows essentially the same features as the integration method in Chapter I.

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[1] Relativistic calculations are given in DYSON [1] (for a scalar particle), KROLL-LAMB [1], FRENCH-WEISSKOPF [1].

[2] FREMBERG [2] Chap. 13, [3] §§ 6, 7.

[3] GUSTAFSON [2], [3].

[4] PAULI [4].

[5] JAUCH [1]. With the α-method this was first established by Pauli (unpublished).

[6] Cf. e.g. CASE [1].

In the ordinary theory, without analytic continuation, a definition of e.g. the field $F_{\mu\nu}(\boldsymbol{x})$ at the electron point $\boldsymbol{z}$ can be obtained in this way. We surround the electron in the local rest system by a sphere (in three dimensions) of radius r and compute the average of $F(\boldsymbol{x})$, let us say $F(\boldsymbol{z}, r)$, over the surface of the sphere; then we go to the limit $r = 0$. Using analytic continuation, we may instead form the expression

$$F(\boldsymbol{z}) = \underset{\alpha=0}{\mathrm{cont}}\ \alpha \int_0^{r_0} r^{\alpha-1} F(\boldsymbol{z}, r)\, dr, \tag{5.2}$$

which, according to the lemma (1.32), coincides with $F(\boldsymbol{z}, 0)$ if $F(\boldsymbol{z}, r)$ is continuous at $r = 0$. If $F(\boldsymbol{z}, r)$ should instead have a pole at $r = 0$ — as is the case for an accelerated electron —, a finite value is defined, which may be regarded as the finite part[1] of the field at $r = 0$, and which coincides with the value given by the Riesz method, Eq. (1.17).

To see this we use the expression derived by Dirac for the field in the vicinity of the world-line: DIRAC [1] Eq. (60). In our present notations, and with $[\boldsymbol{a}, \boldsymbol{b}]_{\mu\nu} = a_\mu b_\nu - a_\nu b_\mu$, we have for the retarded field at a point $\boldsymbol{x} = \boldsymbol{z} + r\,\boldsymbol{e}$, $\boldsymbol{e}$ being a unit vector orthogonal to the world-line at $\boldsymbol{z}$ (i.e. $(\boldsymbol{u}, \boldsymbol{e}) = 0$),

$$\begin{aligned} F_{\mu\nu}(\boldsymbol{x}) = &-\frac{e}{4\pi\varepsilon_0}\{r^{-2}[\boldsymbol{u}, \boldsymbol{e}]_{\mu\nu} + \tfrac{1}{2} r^{-1}[[\dot{\boldsymbol{u}}, \boldsymbol{u}]_{\mu\nu} - (\boldsymbol{e}, \dot{\boldsymbol{u}})[\boldsymbol{u}, \boldsymbol{e}]_{\mu\nu}]\} - \\ &-\frac{e}{4\pi\varepsilon_0}\{\tfrac{1}{8}[3(\boldsymbol{e}, \dot{\boldsymbol{u}})^2 + \dot{\boldsymbol{u}}^2][\boldsymbol{u}, \boldsymbol{e}]_{\mu\nu} - \tfrac{3}{4}(\boldsymbol{e}, \dot{\boldsymbol{u}})[\dot{\boldsymbol{u}}, \boldsymbol{u}]_{\mu\nu} - \tfrac{1}{2}[\ddot{\boldsymbol{u}}, \boldsymbol{e}]_{\mu\nu} - \tfrac{2}{3}[\ddot{\boldsymbol{u}}, \boldsymbol{u}]_{\mu\nu}\} + O(r). \end{aligned} \tag{5.3}$$

In prevision of the continuation (5.2) only the terms independent of r are important, and on averaging over the directions of $\boldsymbol{e}$ we see that all of these drop out except the last. Adding the incident field, we are thus back at the expression (1.17), as we wanted to establish.

It is also possible to give a corresponding definition of energy-momentum quantities. With $\boldsymbol{i}$ another unit vector orthogonal to $\boldsymbol{u}$ and with $\boldsymbol{x}' = \boldsymbol{x} + a\,\boldsymbol{i} = \boldsymbol{z} + r\,\boldsymbol{e} + a\,\boldsymbol{i} = \boldsymbol{z} + r'\,\boldsymbol{e}'$, we form

$$T_{\mu\nu}(\boldsymbol{x}; \boldsymbol{x}') = \varepsilon_0 [F_{\mu\lambda}(\boldsymbol{x}) F_\nu{}^\lambda(\boldsymbol{x}') - \tfrac{1}{4} g_{\mu\nu} F_{\sigma\lambda}(\boldsymbol{x}) F^{\sigma\lambda}(\boldsymbol{x}')]. \tag{5.4}$$

Averaging over the directions of $\boldsymbol{i}$ (for $\boldsymbol{x}$ and a fixed) then yields an expression $T_{\mu\nu}(\boldsymbol{x}, a)$ corresponding to $F(\boldsymbol{z}, r)$ above; in integrals containing $T_{\mu\nu}(\boldsymbol{x}, a)$, a is afterwards to go to nought by analytic continuation as in (5.2).

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[1] Generally, $F(\boldsymbol{z})$ defined by (5.2) equals the coefficient $F_0(\boldsymbol{z})$ in the expansion $F(\boldsymbol{z}, r) = \sum_{-m}^{\infty} F_n(\boldsymbol{z})\, r^n$ (with m an integer); see the lemma (1.32), (1.33), or MA [1] Eqs. (14), (15).

The suitability of this definition can be tested by computing the flow of energy and momentum out through a thin tube Ω, of radius ε, surrounding the world-line and bounded by two planes σ' and σ'' orthogonal to the line, at $\boldsymbol{z}'$ and $\boldsymbol{z}''$:

$$\operatorname*{cont}_{\alpha=0} \alpha \int\limits_0^{a_0} da\, a^{\alpha-1} \int\limits_{\Omega} \frac{\partial\, T_{\mu\nu}(\boldsymbol{x}, a)}{\partial\, x_\nu} (dx)^4. \tag{5.5}$$

Expressing the integral over Ω as a surface integral over the three-dimensional boundary of the region, we can obtain the contribution from the time-like tube surface Σ by performing the analytic continuation under the integral sign, i.e. by putting $a = 0$ in $T_{\mu\nu}$. This contribution is given by Eq. (68) in DIRAC [1], in our notations

$$\int\limits_{\Sigma} T_{\mu\nu}\, e^\nu\, d\Sigma = \int\limits_{s'}^{s''} \left[\frac{e^2}{8\pi\varepsilon_0}\, \varepsilon^{-1}\, \dot{u}_\mu + e\, F_{\mu\nu}(\boldsymbol{z})\, u^\nu\right] ds + O(\varepsilon), \tag{5.6}$$

with $F_{\mu\nu}(\boldsymbol{z})$ defined by (1.17).

In evaluating the integrals over the end surfaces, we need only take those terms in (5.3) into consideration which are proportional to r^{-2} and r^{-1}; thus

$$\begin{aligned} T_{\mu\nu}(\boldsymbol{x}; \boldsymbol{x}')\, u^\nu = \frac{e^2}{32\,\pi^2\,\varepsilon_0}\, u_\mu \Bigg\{ &- \frac{(\boldsymbol{e}, \boldsymbol{e}')}{r^2\, r'^2} + \frac{1}{2\, r^2\, r'} [(\boldsymbol{e}, \dot{\boldsymbol{u}}) + (\boldsymbol{e}, \boldsymbol{e}')(\boldsymbol{e}', \dot{\boldsymbol{u}})] + \\ &+ \frac{1}{2\, r\, r'^2} [(\boldsymbol{e}', \dot{\boldsymbol{u}}) + (\boldsymbol{e}, \boldsymbol{e}')(\boldsymbol{e}, \dot{\boldsymbol{u}})] \Bigg\} + \varepsilon^{-2}\, O(1), \end{aligned} \tag{5.7}$$

where $r'^2 = r^2 + a^2 + 2\, a\, r\, (\boldsymbol{e}, \boldsymbol{i})$. The integrations over the directions of $\boldsymbol{e}$ and $\boldsymbol{i}$ leave only the first term within the braces different from zero. On evaluation we get

$$\begin{aligned} \int\limits_{r \le \varepsilon} T_{\mu\nu}(\boldsymbol{x}, a)\, u^\nu\, d\sigma &= - \frac{e^2}{32\,\pi^2\,\varepsilon_0}\, u_\mu \int\limits_0^{\varepsilon} r^2\, dr \oint \frac{(\boldsymbol{e}, \boldsymbol{e}')}{r^2\, r'^2}\, d\omega + O(\varepsilon) = \\ &= - \frac{e^2}{8\pi\varepsilon_0}\, u_\mu \int\limits_a^{\varepsilon} \frac{dr}{r^2} + O(\varepsilon) = - \frac{e^2}{8\pi\varepsilon_0} \left(\frac{1}{a} - \frac{1}{\varepsilon}\right) u_\mu + O(\varepsilon) \end{aligned} \tag{5.8}$$

(for $a < \varepsilon$), so that the contribution of the end surfaces to the integral in (5.5) becomes

$$\operatorname*{cont}_{\alpha=0} \alpha \int\limits_0^{a_0} da\, a^{\alpha-1} \Big[\int\limits_{\sigma'} - \int\limits_{\sigma''}\Big]\, T_{\mu\nu}(\boldsymbol{x}, a)\, u^\nu\, d\sigma = - \left[\frac{e^2}{8\pi\varepsilon_0}\, \varepsilon^{-1}\, u_\mu\right]_{s'}^{s''} + O(\varepsilon). \tag{5.9}$$

The significant term in (5.9) cancels against the first term on the right in

(5.6); therefore, the total flow of electromagnetic energy and momentum out from the region is given by the second term in (5.6)[1] (it is independent of the radius ε and so can be obtained by letting $\varepsilon \to 0$). When the 'flow' of mechanical energy-momentum is added, we are again led to Eq. (1.19) from the requirement that the total flow should be zero.

A similar procedure could be used in the quantum theory. We should then replace the interaction term in the Lagrangian (2.1) by e.g. $-i\,e\,\psi^\dagger(\boldsymbol{x})\,\gamma^\mu\,\psi(\boldsymbol{x})\cdot$ $\cdot\,A_\mu(\boldsymbol{x}, a)$, the equations of motion being modified accordingly. As a tends to nought by the continuation process of Eq. (5.5), the value zero is obtained in this way too for the self-energy of the electron on the one-electron theory. The advantage of this method is that the commutation properties are not affected, but it has the drawback of not being relativistic for $a \neq 0$, i.e. for $a > 0$ (because $A_\mu(\boldsymbol{x}, a)$ is an average value calculated in a fixed plane $x_0 =$ const.). In that respect the method resembles the procedure of Chapter III; indeed, is is easy to convince oneself that if the self-energy (in the e^2 approximation) is written as a momentum integral, it will again contain a factor g_α of essentially the same form as in (3.35): $g_\alpha = \alpha \int\limits_0^{a_0} a^{\alpha-1} \cos a\,k\;da$ (note that in this section we have not taken the trouble to make the formulas dimensionally correct for $\alpha \neq 0$).

More generally, if a length a is introduced in any way into the formalism, the result of letting it approach zero by analytic continuation (as in Eq. (5.5)) will be to throw away all terms depending on a in expressions which can be expanded in Laurent series in a (with a pole at $a = 0$).

28. Conclusion. In the foregoing pages different aspects of the α-method have been examined, and its suitability for eliminating divergences has been pointed out. We shall now end this investigation with some critical and mostly negative observations. In forming an opinion of the α-method as presented here, we have to allow that it can certainly make no pretence to constitute a definitive form of the theory, at least not in quantum electrodynamics. On the whole, the situation may be said to be more satisfactory in the classical case, where the original formulation of Riesz and Fremberg gives an elegant derivation of results in a consistent and relativistically invariant manner. Of course, the physical content of the results can still be questioned, especially in view of the run-away solutions contained in the equation of motion (1.18); this question is connected with the problem whether the electron is really to be conceived as a point.

Quantum-theoretically matters are complicated as a consequence of the fundamental role played by the commutation relations, which makes it desirable to have the theory in a canonical form. But the application of the integration method in Chapter II is not Hamiltonian; and the Hamiltonian formulation of Chapters III and IV lacks some of the beauty of the Riesz method by not being relativistic (for $\alpha > 0$). This last property will be a general feature of attempts to alter the commutation rules along the lines tried here: by modifying the D function appearing in the commutators. For if this function is represented as a Fourier integral in momentum space over the light-cone mantle (or a hyperboloid surface), what is needed in order to eliminate divergences is a modification for infinitely large momenta k or k_0; and, the only invariant formed from the vector $\boldsymbol{k}$ alone being a function of $(\boldsymbol{k}, \boldsymbol{k})$, which is constant on any invariant surface, it is not possible to devise a modification that is at the same time effectual and relativistic. If an extraneous vector is introduced, as the vector $\boldsymbol{l}$ in Chapter III – where $(\boldsymbol{k}, \boldsymbol{k})$ was replaced by $(\boldsymbol{k}, \boldsymbol{l})$ –, it has no physical significance.

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[1] The proof of the corresponding formula in the case of the Riesz field, Eq. (1.16), could be conducted in a similar way. With the integral of $\partial\, T^{(\alpha)}_{\mu\nu} / \partial\, x_\nu$ expressed as a surface integral, the contribution from Σ is again obtained in the form (5.6) for $\alpha = 0$, and it remains to show that the analytic continuation of the end-surface integrals will just cancel the first term in (5.6) (except for terms vanishing with ε). This can at least be made plausible as follows. Our result (1.23) in section 3 shows that these integrals are of the required form (5.9) if u_μ is constant; and if u_μ is not constant, the significant additive terms are seen from (5.7) (with $\boldsymbol{e}' = \boldsymbol{e}$, $r' = r$) to vanish on letting α go to nought under the integral sign (the integrand contains the factor $(\boldsymbol{x} - \boldsymbol{z}, \dot{\boldsymbol{u}})/r^4$, which gives the result zero when integrated over the directions).

From a relativistic point of view it would therefore be better to make the change not in the vacuum fields but in the interaction term, as is done in the Riesz-Gustafson method.

Even if the non-relativistic character of the canonical formalism can be done away with in the end results by the procedure of section 25, it is not very satisfactory to associate the purposely invariant formulation of Schwinger with the non-invariant procedure here. However, Schwinger's method should be supplemented by some convergence-producing formalism, and then the α-method offers one possibility. The physically observable predictions of the theory will not differ from those of Schwinger's theory, provided the original divergences have been removed by the renormalization procedure. Now it seems that not all the infinities one meets with are of this kind; thus in the case of vector mesons logarithmic divergences remain even after renormalization.[1] The application of the α-method in such instances is subject to some ambiguity because of the undetermined quantity l_0 entering into the results. It was suggested in section 24, Eq. (4.34), that l_0 should be set equal to the Compton wave-length $\hbar/mc$, but at present we have no principle that would allow us to determine the value theoretically. Another question which is not cleared up is that of the divergences in higher approximations.

Though it is thus evident that the theory is still incomplete, the α-method is of interest as indicating one possible way out of the divergence difficulties. Whether that way is physically justified, and so whether, in a future theory, it will be found convenient to resort to analytic continuation in some form, is not clear.

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[1] See CASE [2].

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Institutionen för mekanik och matematisk fysik, Lund.

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